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Equivalent uniform moment factors for lateral–torsional buckling of steel members

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Abstract

Steelwork design standards consider lateral–torsional buckling as one of the ultimate limit states that must be checked for steel members in bending. The buckling resistance assessment is usually based on buckling curves and requires the computation of the elastic critical moment, which is strongly dependent on both the bending moment distribution and restrictions at end supports. This paper focuses on the equivalent uniform moment factor (EUMF) which is used to compute the elastic critical moment. A review of EUMF values given by modern steelwork standards and their comparison with recent results presented in the literature shows that whilst codes may lead to very conservative values for simply supported beams, non-conservative values are obtained in the case of support types designed to restrict lateral bending and warping. In order to clarify situations where EUMF values proposed by modern codes appear contradictory with recent computational results, the paper presents a significant set of EUMF values obtained using both finite difference and finite element techniques. Particular attention has been given to instances where lateral bending and warping are prevented at beam supports since very few results from these cases have been published. A major advantage of codes, such as the American AISC LRFD and the British BS 5950-1, is that they provide closed-form expressions to compute the EUMF for any moment distribution. Unfortunately, these closed-form expressions do not take into account changes in the EUMF due to end support restrictions. This paper presents a general closed-form expression that not only delivers similar advantages but also improves the results given by those codes for some loading cases.

Keywords: Elastic lateral–torsional buckling; Equivalent uniform moment factor; Moment gradient factor; End restrictions; Steel members; Steelworks Standards

1. Introduction

Structural stability has always been a key point in the design of steel structures. With the increasing use of computer programs in the design process of structures, a clear understanding and appropriate checking of member stability are becoming even more relevant. Design standards and codes usually refer to three main situations: members in compression, members in bending, and members in combined bending and compression. Depending on flexural slenderness, torsional rigidity and geometry of the cross-section, members in compression may suffer one of the following buckling types: flexural buckling, torsional buckling or flexural–torsional buckling. Members undergoing bending with respect to the major axis of their section may develop lateral–torsional buckling, which implies bending about the minor axis (lateral bending) and torsion of the section. Finally, in members subject to bending and compression (usually referred to as beam–columns), the axial compression amplifies the bending moment. This second-order effect is usually taken into consideration by means of interaction factors. In addition to presenting a global picture of stability-related problems, it is necessary to refer to distortional buckling and local buckling, which have special significance for non-compact sections.

Because of its importance in the design of beams and beam–columns [16], lateral–torsional buckling continues to attract the attention of many researchers. Solutions for simple cases may be obtained from traditionally accepted books such as [26,6,27]. A summary of research performed before the computer era may be found in the renowned paper by Clark and Hill [7], which was intended as background material for the design of beams whose strength is controlled by lateral–torsional buckling. With the advent of computer-aided

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numerical techniques, such as finite differences and finite elements, new solutions to complex lateral–torsional buckling problems were presented. Summary papers that give quite precise panoramas on the research performed at the time are those written by Nethercot [18,19].

Research developments have been followed by the realization of updated design codes and standards. Based on the limit state concept, modern steel structures codes, such as AISC LRFD [1,2], BS 5950-1 [5] and EC3 [9], provide design procedures to assess the lateral–torsional buckling resistance of beams. As a first step, these procedures generally require the determination of the elastic critical buckling moment. Initial imperfections, residual stresses and inelastic buckling are taken into account through the use of buckling curves [28].

The elastic critical moment is directly affected by the following factors [15]: material properties, such as the modulus of elasticity and shear modulus; geometric properties of the cross-section, such as the torsion constant, warping constant, and moment of inertia about the minor axis; properties of the beam, such as length of the beam, and lateral bending and warping restrictions at supports; and, finally, loading, since lateral–torsional buckling is greatly affected by moment diagram and loading position with respect to the section shear centre. As a theoretically based and generally accepted procedure, consideration of the bending moment diagram is taken into account by means of the equivalent uniform moment factor, also called the moment gradient correction factor. The elastic critical moment of the simply supported beam with uniform moment is multiplied by this factor to obtain the elastic critical moment for any bending moment diagram.

In spite of the extensive research reported in the literature, some important discrepancies on the equivalent uniform moment factor still remain unresolved. A major comprehensive effort to offer a general procedure to determine the elastic critical moment of beams was presented by Nethercot and Rockey [17]. In their work, Nethercot and Rockey considered four types of support (simply supported, warping fixed, lateral bending fixed and completely fixed) and three loading levels (top flange, shear center and bottom flange) resulting in 41 different cases. More recently, Suryoatmono and Ho [25] have reported the results of an elaborate study using a finite difference technique to solve the governing differential equation. They have shown how equivalent uniform moment factors provided by AISC LRFD [1,2] are very conservative for most moment diagrams but may be non-conservative in particular cases. Their paper provides closed-form expressions which match the numerical results of each of the loading cases presented. However, this study does not consider lateral bending and warping prevention as situations that need a specific analysis. Lim et al. [14], using finite elements, investigated linear moment diagrams with three different end support restraints: simply supported, warping prevented, and both lateral bending and warping prevented. In their paper they proposed new equations to be used in determining the moment gradient correction factor for those particular cases. Finally, in the context of the ECCS-Validation Group, which is in charge of reviewing specifications for members in bending to be used in the Eurocode for steel structures, Greiner et al. [11] and Salzgeber [24] provide equivalent uniform factors for a very complete set of moment diagrams. Results for monosymmetric beams may be found in the work of Baláz and Koleková [3] and Helwig et al. [12]. Moreover, beams with top bracing and bottom bracing have been considered in Park and Kang [21], Park et al. [22] and Greiner et al. [11].

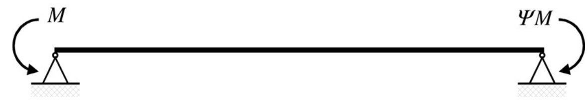

Fig. 1. Linear moment distribution.

This paper focus on the equivalent uniform moment factor for beams with double symmetric steel sections which have the load acting at the shear centre of the section and considers beams subject to general moment diagrams with different end support conditions. Bracing for torsion is assumed at the end supports. A summary of the procedures used by the three standards and codes referred to above is presented first, i.e., the American AISC LRFD, the British BS5950-1 and the emerging Eurocode 3. Recent numerical results are then presented, with the goal of pointing out possible discrepancies with design codes and cases which need further consideration. Subsequently, new results based on both finite differences and finite elements are presented in order to clarify those discrepancies. Finally, a new general closed-form expression is proposed for determining the equivalent uniform moment factor for any moment distribution and end support conditions.

2. Design code procedures

For double symmetric beams loaded at section shear centre, the value of M_{cr} may be computed by the expression

$$M_{cr} = C_1 \frac{\pi^2 E I_z}{(kL)^2} \left\{ \sqrt{\left(\frac{k}{k_w}\right)^2 \frac{I_w}{I_z} + \frac{(kL)^2 G I_t}{\pi^2 E I_z}} \right\} \tag{1}$$

where the support conditions are introduced by means of the lateral bending coefficient k and the warping coefficient k_w, whose values are 1 for free bending and free warping, and 0.5 for prevented bending and prevented warping. The moment gradient along the beam is taken into account by the coefficient C_1 which is also affected by the lateral bending conditions at end supports.

In this section we present a schematic overview of the procedures proposed by some steelwork standards to determine the elastic critical moment for lateral–torsional buckling and the values they give for the moment gradient factor.

2.1. AISC LRFD

AISC LRFD [1,2] specifications define the nominal elastic lateral–torsional buckling moment as

$$M_{cr} = C_b \frac{\pi}{L_b} \sqrt{E I_z G I_t + \left(\frac{\pi E}{L_b}\right)^2 I_z I_w} \tag{2}$$

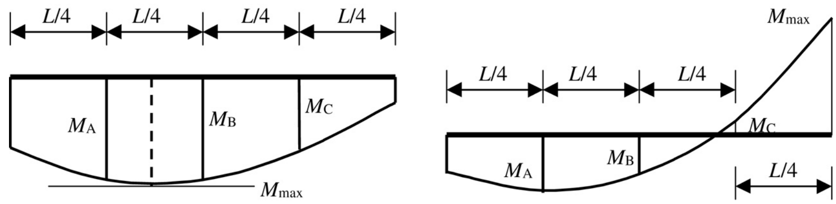

Fig. 2. Moment diagrams and moment values for Eq. (4).

where L_b is the lateral buckling length and C_b is the moment gradient coefficient.

Eqs. (1) and (2) are identical when the lateral bending coefficient k and the warping coefficient k_w have the same value, i.e., when the beam has the same degree of prevention for lateral bending as for warping at each of its supports. When the lateral buckling length L_b is made equal to kL, and k equal to k_w, both expressions are identical. Therefore, coefficients C_b and C_1 are one and the same.

Following the research work performed by Salvadory [23], values for C_b were given in the AISC LRFD [1] for the case of linear moment distribution (Fig. 1) by the following closed-form expression:

$$C_b = 1.75 - 1.05\psi + 0.3\psi^2 \qquad \text{with } C_b \leq 2.3. \tag{3}$$

The latest edition of AISC LRFD [2] incorporates, with very small modifications, the closed-form expression proposed by Kirby and Nethercot [13] which is valid for any moment distribution. With reference to Fig. 2, the nonlinear moment gradient coefficient is given by

$$C_b = \frac{12.5M_{max}}{2.5M_{max} + 3M_A + 4M_B + 3M_C} \tag{4}$$

where M_{max} is the maximum moment, and M_A, M_B and M_C are the values of the moment at $L/4$, $L/2$ and $3L/4$ of the length respectively. All the moments in Eq. (4) are absolute values, i.e., positive values regardless of the sign of the bending moment.

It should be noted that AISC LRFD does not consider the possibility of k being different from k_w. Moreover, values of C_b for any support condition are the same regardless of the value of lateral bending and warping coefficients, k and k_w respectively.

2.2. BS 5950

The 2000 edition of the British code for steelworks in buildings [5] incorporates a formulation very similar to that of AISC LRFD [2]. The BS 5950-1 equivalent uniform moment factor, called m_{LT}, is the inverse of factors C_1 (Eq. (1)) and C_b (Eq. (2)). The values of m_{LT} and the corresponding C_1 are

$$m_{LT} = 0.2 + \frac{0.15M_A + 0.5M_B + 0.15M_C}{M_{max}} \quad \text{but } m_{LT} \geq 0.44 \tag{5}$$

$$C_1 = \frac{M_{max}}{0.2M_{max} + 0.15M_A + 0.5M_B + 0.15M_C} \quad \text{but } C_1 \leq 2.273. \tag{6}$$

Values of M_{max}, M_A, M_B and M_C are absolute values and are given in Fig. 2. Multiplying Eq. (6), top and bottom, by 12.5 we may see more clearly the similarity with the moment distribution factor given by AISC LRFD [2]:

$$C_1 = \frac{12.5M_{max}}{2.5M_{max} + 1.875M_A + 6.25M_B + 1.875M_C}. \tag{7}$$

Nevertheless, the two formulations differ in the weight coefficient given to the midspan moment, M_B, and, consequently, to M_A and M_C.

2.3. Eurocode 3

Lateral–torsional buckling of beams is considered by Eurocode 3 [9] as an ultimate limit state related to member buckling resistance. The buckling resistance is obtained by multiplying the resistance of the cross-section by a reduction factor χ_{LT}. This reduction factor is a function of two other parameters: the imperfection factor α and the non-dimensional slenderness $\bar{\lambda}_{LT}$. The parameter α takes into account the initial member imperfection, residual stresses and other nonlinear effects. Given the cross-section and the steel grade of the member, EC3 [9] provides the coefficient α, which may adopt four values for members in bending. The non-dimensional slenderness $\bar{\lambda}_{LT}$ is obtained by the expression

$$\bar{\lambda}_{LT} = \sqrt{\frac{M_R}{M_{cr}}} \tag{8}$$

where M_R is the bending resistance moment of the cross-section and M_{cr} is the elastic critical moment for lateral–torsional buckling. The last available draft of Eurocode 3 [9] does not provide information on how to compute M_{cr}.

3. Recently reported results

Even though lateral–torsional buckling has been studied for more than a century now, only with the use of numerical techniques such as finite differences and finite elements has it been possible to perform extensive research. An overview of numerical results that have been reported in the literature recently is presented below.

Table 1
Lim et al. [14]

k	k_w	S_1	S_2
1.0	1.0	1.00	0.16
0.5	0.5	1.00	0.18
1.0	0.5	0.80	0.10

S_1 and S_2 values (Eq. (9)).

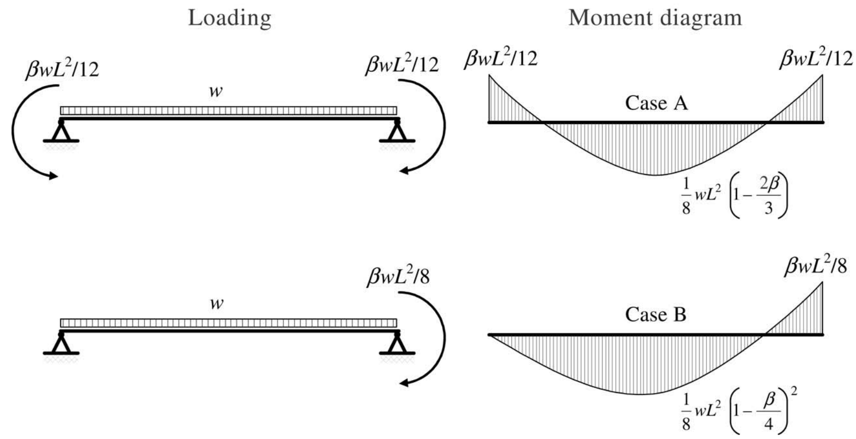

Fig. 3. Loading and moment diagram cases.

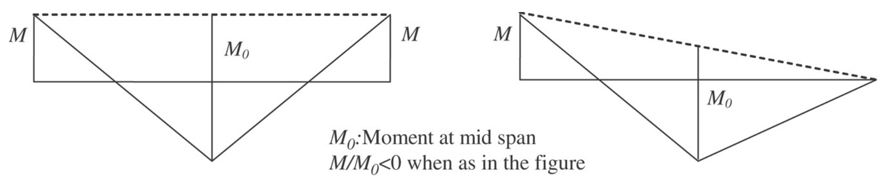

Fig. 4. Triangular moment diagrams.

3.1. R. Greiner and co-workers

Prof. R. Greiner (TU Graz), as member of the ECCS Working Group No. 8, lead by Prof. J. Linder (TU Berlin), has undertaken extensive research on beam–column stability in the last few years. His research has provided the theoretical background for the buckling resistance of members in EC3 [9]. Regarding the computation of the elastic critical moment, Greiner and Lindner [10] and Greiner et al. [11] presented a quite complete numerical study on lateral torsional buckling of beams, using finite element techniques, and provide a complete set of values for the moment diagram factor. Their results are presented both in the form of diagrams and in the form of numerical tables. In their work, Greiner et al. [11] also present results for beams with upper flange and lower flange restraints.

3.2. Suryoatmono and Ho

Using a finite difference approach to compute elastic critical moments for wide flange sections, Suryoatmono and Ho [25] have recently proposed closed-form expressions to obtain the moment gradient factor for some linear and nonlinear cases. Since there is no reference to lateral bending and warping restrictions it must be assumed that the proposed expressions are considered valid for all support situations, as in AISC LRFD and BS 5950-1 procedures. Of course, restrictions to lateral bending and warping at supports should be taken into account through appropriate lateral buckling length.

Besides linear moment distribution and concentrated load, Suryoatmono and Ho [25] considered two nonlinear moment distributions (Fig. 3) and gave specific closed-form expressions for each loading case. Their closed-form expressions are the result of applying a polynomial of degree 6 to the numerical results.

3.3. N.-H. Lim, N.-H. Park, Y.-J. Kang and I.-H. Sung

Using both the Bubnov–Galerkin approach [29] and the Finite Element Method [4], a research team from South Korea [14] has conducted an extensive investigation into elastic buckling in I-beams. Even though their study focuses on linear moment diagrams only, lateral bending prevention and warping prevention are also considered.

The elastic critical moment is computed using a formulation identical to that given in Eq. (1). The lateral bending coefficient k and warping coefficient k_w are used in a similar way, with only differences of nomenclature. The equivalent uniform moment factor C_1 is determined using the following expression:

$$C_1 = \frac{2}{\sqrt{S_1}\sqrt{(1+\psi)^2 + S_2(1-\psi)^2}} \tag{9}$$

where coefficients S_1 and S_2 are given in Table 1 for different support conditions.

4. New numerical results

As indicated before, in order to complete the numerical results available in the literature, the equivalent uniform moment factor has been obtained for beams with a quite extensive set of loadings and end support conditions. First, working with finite elements general purpose software [8] we have studied the lateral–torsional buckling of an IPE500 cross-section beam for two length cases: 8 m and 16 m. The goal of this analysis was to confirm that coefficient C_1 varies significantly not only with moment distribution but also with end support conditions. Once the finite elements results confirmed that variation, finite differences specific purpose software [20] was used to integrate the governing differential

Table 2
C_1 values for linear moment distribution and for concentrated load (Finite Elements)

	Linear moment distribution			
	$k = 1, k_w = 1$		$k = 0.5, k_w = 0.5$	
	IPE500		IPE500	
ψ	$L = 8$	$L = 16$	$L = 8$	$L = 16$
1.00	1.000	1.000	1.000	1.000
0.90	1.052	1.052	1.052	1.052
0.80	1.110	1.109	1.110	1.110
0.75	1.140	1.140	1.140	1.140
0.70	1.173	1.172	1.172	1.173
0.60	1.242	1.241	1.241	1.242
0.50	1.319	1.316	1.317	1.318
0.40	1.403	1.399	1.401	1.402
0.30	1.496	1.489	1.492	1.494
0.25	1.545	1.537	1.541	1.544
0.20	1.597	1.587	1.591	1.595
0.10	1.709	1.694	1.699	1.705
0.00	1.830	1.809	1.815	1.824
−0.10	1.959	1.931	1.936	1.950
−0.20	2.097	2.059	2.061	2.082
−0.25	2.167	2.125	2.123	2.149
−0.30	2.239	2.192	2.184	2.216
−0.40	2.381	2.327	2.299	2.345
−0.50	2.517	2.459	2.399	2.462
−0.60	2.637	2.582	2.472	2.557
−0.70	2.727	2.685	2.508	2.616
−0.75	2.755	2.724	2.509	2.627
−0.80	2.768	2.750	2.498	2.626
−0.90	2.742	2.750	2.439	2.578
−1.00	2.642	2.661	2.339	2.477

	Concentrated load and two moments			
	$k = 1, k_w = 1$		$k = 0.5, k_w = 0.5$	
	IPE500		IPE500	
M/M_0	$L = 8$	$L = 16$	$L = 8$	$L = 16$
2.0	1.113	1.112	1.036	1.032
1.0	1.174	1.174	1.043	1.045
0.5	1.236	1.240	1.043	1.055
0.0	1.358	1.380	1.027	1.070
−0.5	1.514	1.742	0.880	1.040
−0.6	2.092	2.600	1.184	1.486
−0.7	2.533	3.235	1.521	2.058
−1.0	2.347	2.569	2.276	3.281
−1.5	1.729	1.775	2.028	2.253
−2.0	1.482	1.501	1.693	1.755

	Concentrated load and one moment			
	$k = 1, k_w = 1$		$k = 0.5, k_w = 0.5$	
	IPE500		IPE500	
M/M_0	$L = 8$	$L = 16$	$L = 8$	$L = 16$
2.0	1.137	1.135	1.016	1.020
1.0	1.217	1.220	1.031	1.045
0.5	1.278	1.287	1.035	1.058
0.0	1.358	1.380	1.027	1.070
−0.5	1.435	1.497	0.980	1.061
−0.6	1.441	1.518	0.962	1.053
−0.7	1.548	1.652	1.011	1.121
−1.0	2.622	2.958	1.644	1.923
−1.5	3.421	3.638	2.621	3.253
−2.0	3.160	3.198	2.950	3.446

equations for all possible end support conditions. In what follows the paper presents results for both a narrow flange profile (IPE500) and a wide flange profile (HEB500).

Table 3
C_1 coefficients for uniform loading (Finite Elements)

	Uniform loading and two moments			
	$k = 1, k_w = 1$		$k = 0.5, k_w = 0.5$	
	IPE500		IPE500	
β	$L = 8$	$L = 16$	$L = 8$	$L = 16$
0.00	1.147	1.148	0.975	0.977
0.50	1.203	1.217	0.939	0.958
0.75	1.236	1.277	0.895	0.934
0.90	1.858	1.983	1.268	1.359
1.00	2.421	2.688	1.587	1.751
1.05	2.744	3.136	1.771	1.993
1.20	3.686	4.646	2.415	2.974
1.35	4.006	4.946	3.100	4.351
1.50	3.700	4.262	3.602	5.518
2.00	2.485	2.597	3.223	3.760
75.00	1.017	1.017	1.028	1.020

	Uniform loading and one moment			
	$k = 1, k_w = 1$		$k = 0.5, k_w = 0.5$	
	IPE500		IPE500	
β	$L = 8$	$L = 16$	$L = 8$	$L = 16$
0.00	1.147	1.148	0.975	0.977
0.50	1.196	1.204	0.964	0.976
0.75	1.397	1.416	1.087	1.113
0.90	1.869	1.909	1.421	1.472
1.00	2.230	2.296	1.671	1.746
1.05	2.422	2.507	1.803	1.895
1.20	3.028	3.201	2.221	2.385
1.35	3.592	3.897	2.648	2.928
1.50	3.961	4.330	3.032	3.469
2.00	3.860	3.954	3.590	4.248
75.00	1.855	1.840	1.866	1.867

4.1. Finite Elements results

The finite element analysis has been performed using shell elements to model the beam. Accordingly, the IPE 500 cross-section has been simplified to an I-beam section made up of three plates. Results with this technique have been obtained for both simply supported ends ($k = k_w = 1$) and fixed ends ($k = k_w = 0.5$). Tables 2 and 3 give the C_1 values for the following moment distributions: linear moment distribution (Fig. 1); concentrated load with both two equal end moments and one end moment (Fig. 4); and uniform loading with both two equal end moments and one end moment (Fig. 3).

Eq. (1) gives the elastic critical moment as a function of C_1, k and k_w. Therefore, to obtain C_1 from the elastic critical moment computed using numerical techniques it is necessary to decouple the effects of each coefficient. When k is equal to 0.5, the critical moment is multiplied by a factor of two. The effect of k_w not being equal to 1 is equivalent to multiplying the critical moment by the following factor:

$$\frac{\sqrt{1 + \frac{\pi^2 E I_w}{k_w^2 L^2 G I_t}}}{\sqrt{1 + \frac{\pi^2 E I_w}{L^2 G I_t}}}. \tag{10}$$

To obtain the values given in Tables 2 and 3, this last effect is computed using the following section properties for

Table 4
C_1 values for linear moment distribution (Finite Differences)

	Finite Differences results															
	$k=1, k_w=1$				$k=1, k_w=0.5$				$k=0.5, k_w=1$				$k=0.5, k_w=0.5$			
	IPE500		HEB500		IPE500		HEB500		IPE500		HEB500		IPE500		HEB500	
ψ	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$
1.00	1.000	1.000	1.000	1.000	1.122	1.107	1.121	1.103	1.109	1.073	1.105	1.067	1.000	1.000	1.000	1.000
0.90	1.052	1.052	1.052	1.052	1.181	1.164	1.180	1.161	1.167	1.129	1.163	1.123	1.050	1.050	1.050	1.050
0.80	1.110	1.109	1.110	1.109	1.246	1.228	1.244	1.224	1.230	1.190	1.226	1.184	1.107	1.107	1.106	1.108
0.75	1.141	1.140	1.140	1.140	1.281	1.262	1.279	1.258	1.264	1.223	1.260	1.216	1.138	1.138	1.138	1.138
0.70	1.173	1.172	1.173	1.172	1.317	1.298	1.315	1.294	1.299	1.257	1.295	1.250	1.170	1.170	1.170	1.170
0.60	1.242	1.240	1.242	1.240	1.396	1.375	1.394	1.371	1.375	1.331	1.371	1.323	1.240	1.239	1.240	1.239
0.50	1.319	1.315	1.318	1.315	1.484	1.460	1.482	1.455	1.458	1.411	1.454	1.403	1.316	1.315	1.316	1.315
0.40	1.403	1.398	1.402	1.397	1.581	1.554	1.578	1.549	1.549	1.499	1.544	1.490	1.401	1.398	1.401	1.398
0.30	1.496	1.487	1.495	1.486	1.689	1.658	1.686	1.652	1.648	1.594	1.642	1.584	1.494	1.490	1.494	1.489
0.25	1.546	1.535	1.544	1.534	1.748	1.715	1.745	1.708	1.700	1.644	1.695	1.634	1.544	1.539	1.544	1.538
0.20	1.598	1.585	1.596	1.583	1.809	1.773	1.806	1.766	1.755	1.697	1.749	1.686	1.597	1.590	1.596	1.589
0.10	1.710	1.690	1.707	1.687	1.944	1.900	1.940	1.891	1.869	1.806	1.863	1.796	1.709	1.699	1.708	1.697
0.00	1.831	1.803	1.828	1.799	2.092	2.039	2.088	2.029	1.989	1.922	1.982	1.910	1.831	1.816	1.830	1.813
−0.10	1.962	1.923	1.957	1.918	2.257	2.191	2.251	2.178	2.112	2.042	2.105	2.029	1.962	1.940	1.960	1.936
−0.20	2.101	2.050	2.095	2.043	2.437	2.355	2.429	2.339	2.235	2.161	2.228	2.148	2.100	2.069	2.097	2.063
−0.25	2.173	2.114	2.166	2.107	2.533	2.441	2.524	2.424	2.294	2.219	2.287	2.206	2.170	2.135	2.167	2.128
−0.30	2.246	2.180	2.238	2.172	2.632	2.530	2.622	2.511	2.351	2.275	2.343	2.262	2.241	2.200	2.237	2.192
−0.40	2.393	2.312	2.383	2.302	2.840	2.715	2.828	2.691	2.452	2.378	2.444	2.365	2.378	2.326	2.374	2.316
−0.50	2.536	2.443	2.524	2.430	3.056	2.905	3.041	2.877	2.529	2.461	2.522	2.448	2.504	2.440	2.498	2.428
−0.60	2.666	2.565	2.654	2.551	3.272	3.095	3.255	3.062	2.573	2.515	2.567	2.504	2.606	2.533	2.599	2.519
−0.70	2.769	2.670	2.757	2.655	3.474	3.273	3.455	3.235	2.577	2.532	2.573	2.523	2.669	2.590	2.662	2.575
−0.75	2.804	2.710	2.793	2.697	3.561	3.351	3.541	3.311	2.563	2.524	2.560	2.517	2.682	2.602	2.674	2.586
−0.80	2.824	2.739	2.815	2.726	3.633	3.417	3.612	3.376	2.539	2.506	2.536	2.499	2.680	2.600	2.672	2.585
−0.90	2.809	2.745	2.802	2.734	3.696	3.481	3.676	3.438	2.462	2.438	2.460	2.433	2.631	2.553	2.624	2.538
−1.00	2.711	2.659	2.706	2.650	3.602	3.396	3.583	3.356	2.356	2.335	2.354	2.330	2.527	2.453	2.521	2.438

the simplified IPE500 section: $I_z = 2.138\text{e}{-5}\ \text{m}^4$; $I_t = 7.23\text{e}{-7}\ \text{m}^4$; $I_w = 1.336\text{e}{-6}\ \text{m}^6$.

From the finite elements results the following two main conclusions may be drawn. First, as Nethercot and Rockey [17] and other authors have already pointed out, coefficient C_1 varies with the length of the beam. This variation is less than 2.5% for the linear moment distribution but increases significantly for other loading cases. Second, the equivalent uniform moment factor is coupled to lateral bending and warping conditions at supports. The equations given by some standards, such as AISC LRFD [2] and BS 5950-1 [5], to compute the elastic critical moment assume that coefficient C_1 does not vary with end support conditions, which only serve to change the buckling length of the beam. From the finite elements results it is clear that C_1 values for beams with lateral bending and warping prevented at end supports ($k = k_w = 0.5$) may be significantly lower that those of simply supported beams. In any case, these results show the need for a deeper analysis on the variation of equivalent uniform moment factor values due to lateral bending and warping prevention at supports.

4.2. *Finite Differences results*

With the nomenclature used in EC3 [9], where (x–x) is the axis along the member, (y–y) is the major axis of cross-section and (z–z) is the minor axis of the cross-section, the governing differential equation for the lateral torsional buckling is

$$E I_w \frac{d^4\phi}{dx^4} - G I_t \frac{d^2\phi}{dx^2} - \frac{1}{E I_z} M_y^2 \phi + \frac{1}{E I_z} M_y M_z = 0 \qquad (11)$$

with

$$\frac{dM_y}{dx} = V_z \qquad \frac{dV_z}{dx} = -q_z$$
$$\frac{dM_z}{dx} = -V_y \qquad \frac{dV_y}{dx} = 0 \qquad (12)$$

where q_z is the distributed load acting on the beam, V_y and V_z are the shear forces, M_y and M_z are the bending moments, and ϕ is torsion deformation. In order to be able to impose appropriate boundary conditions at supports, the internal shear forces and the bending moment components in Eqs. (11) and (12) are referred to the axis in the undeformed configuration.

Rolled section profiles have been used in the analysis performed using a finite difference approach to solve the governing differential equations. In order to cover both small flange sections and large flange sections, as well as both small and large values of R [17] two sections, IPE 500 and HEB 500, and two beam lengths, 8 m and 16 m, have been used. For each of these four cases all of the different support conditions have been considered: lateral bending free ($k = 1$) or prevented ($k = 0.5$); warping free ($k_w = 1$) or warping prevented ($k_w = 0.5$). Tables 4–8 and Figs. 5–9 show the new results for the three cases studied: linear moment distribution, uniform loading and concentrated load.

For the case of linear moment distribution, Fig. 5 and Table 4 show that the values of C_1 for beams with prevented both lateral bending and warping at supports are lower than those for simply supported beams. AISC LRFD [2] is conservative for all cases.

Table 5
C_1 values for uniform loading and two end moments (Finite Differences)

	Finite Differences results															
	$k = 1, k_w = 1$				$k = 1, k_w = 0.5$				$k = 0.5, k_w = 1$				$k = 0.5, k_w = 0.5$			
	IPE500		HEB500		IPE500		HEB500		IPE500		HEB500		IPE500		HEB500	
β	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$
0.00	1.131	1.130	1.131	1.129	1.223	1.212	1.222	1.209	0.990	0.976	0.989	0.973	0.967	0.966	0.967	0.966
0.50	1.192	1.189	1.192	1.188	1.272	1.262	1.271	1.260	0.929	0.920	0.928	0.919	0.946	0.943	0.946	0.943
0.75	1.244	1.240	1.243	1.239	1.317	1.308	1.316	1.306	0.869	0.864	0.868	0.863	0.922	0.917	0.921	0.915
0.90	1.924	1.916	1.923	1.915	2.036	2.023	2.035	2.020	1.217	1.213	1.217	1.213	1.342	1.330	1.342	1.328
1.00	2.605	2.596	2.604	2.595	2.782	2.764	2.780	2.759	1.516	1.512	1.516	1.511	1.733	1.711	1.731	1.705
1.05	3.042	3.035	3.042	3.033	3.287	3.264	3.285	3.259	1.691	1.686	1.691	1.685	1.975	1.945	1.972	1.937
1.20	4.685	4.648	4.682	4.640	5.702	5.600	5.694	5.575	2.344	2.324	2.342	2.320	2.981	2.892	2.974	2.872
1.35	5.431	5.181	5.405	5.138	8.321	7.568	8.252	7.419	3.210	3.117	3.200	3.100	4.539	4.248	4.513	4.189
1.50	4.703	4.498	4.682	4.464	7.066	6.402	7.002	6.275	4.188	3.901	4.157	3.853	6.130	5.468	6.066	5.344
2.00	2.703	2.680	2.701	2.676	3.382	3.266	3.372	3.241	4.758	4.505	4.744	4.375	3.899	3.857	3.896	3.846
75.00	1.018	1.018	1.018	1.018	1.143	1.127	1.142	1.123	1.133	1.097	1.130	1.090	1.018	1.019	1.018	1.019

Table 6
C_1 values for uniform loading and one end moment (Finite Differences)

	Finite Differences results															
	$k = 1, k_w = 1$				$k = 1, k_w = 0.5$				$k = 0.5, k_w = 1$				$k = 0.5, k_w = 0.5$			
	IPE500		HEB500		IPE500		HEB500		IPE500		HEB500		IPE500		HEB500	
□	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$
0.00	1.131	1.130	1.131	1.129	1.223	1.212	1.222	1.209	0.990	0.976	0.989	0.973	0.967	0.966	0.967	0.966
0.50	1.182	1.179	1.182	1.178	1.271	1.259	1.270	1.256	0.960	0.950	0.959	0.948	0.965	0.963	0.965	0.962
0.75	1.390	1.383	1.389	1.382	1.496	1.480	1.494	1.476	1.071	1.061	1.070	1.059	1.102	1.096	1.101	1.095
0.90	1.873	1.861	1.872	1.860	2.027	2.002	2.024	1.996	1.391	1.380	1.390	1.378	1.457	1.446	1.456	1.444
1.00	2.252	2.238	2.251	2.236	2.453	2.419	2.450	2.412	1.628	1.617	1.627	1.614	1.730	1.714	1.729	1.711
1.05	2.459	2.443	2.457	2.441	2.691	2.651	2.687	2.643	1.754	1.742	1.753	1.739	1.878	1.859	1.876	1.855
1.20	3.142	3.124	3.140	3.121	3.519	3.456	3.513	3.442	2.157	2.142	2.156	2.139	2.369	2.336	2.366	2.329
1.35	3.854	3.829	3.851	3.825	4.517	4.412	4.508	4.390	2.586	2.562	2.583	2.558	2.919	2.863	2.915	2.851
1.50	4.405	4.312	4.396	4.295	5.563	5.361	5.546	5.316	3.011	2.970	3.007	2.962	3.485	3.390	3.477	3.370
2.00	4.276	3.986	4.240	3.946	5.703	5.206	5.653	5.116	3.936	3.757	3.915	3.729	4.427	4.178	4.403	4.130
75.00	1.867	1.837	1.863	1.833	2.137	2.082	2.132	2.070	2.036	1.967	2.029	1.955	1.874	1.857	1.872	1.854

Table 7
C_1 values for concentrated load and two end moments (Finite Differences)

	Finite Differences results															
	$k = 1, k_w = 1$				$k = 1, k_w = 0.5$				$k = 0.5, k_w = 1$				$k = 0.5, k_w = 0.5$			
	IPE500		HEB500		IPE500		HEB500		IPE500		HEB500		IPE500		HEB500	
M/M_0	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$	$L = 8$	$L = 16$
2.0	1.106	1.106	1.106	1.106	1.219	1.205	1.218	1.202	1.112	1.082	1.109	1.077	1.026	1.026	1.026	1.026
1.0	1.165	1.164	1.165	1.164	1.271	1.258	1.270	1.256	1.108	1.082	1.106	1.077	1.038	1.037	1.038	1.037
0.5	1.227	1.225	1.227	1.225	1.326	1.314	1.325	1.312	1.101	1.078	1.099	1.074	1.048	1.047	1.047	1.047
0.0	1.359	1.355	1.393	1.354	1.442	1.432	1.441	1.430	1.078	1.061	1.076	1.058	1.061	1.059	1.061	1.058
−0.5	1.713	1.702	1.712	1.700	1.794	1.784	1.793	1.781	0.950	0.946	0.950	0.945	1.044	1.033	1.043	1.031
−0.6	2.591	2.583	2.590	2.581	2.854	2.828	2.852	2.821	1.310	1.306	1.310	1.305	1.508	1.484	1.506	1.478
−0.7	3.354	3.316	3.350	3.308	4.248	4.119	4.237	4.090	1.752	1.743	1.751	1.742	2.129	2.074	2.125	2.062
−1.0	2.711	2.659	2.706	2.650	3.602	3.396	3.583	3.356	3.085	2.966	3.073	2.944	3.590	3.406	3.574	3.367
−1.5	1.815	1.807	1.814	1.806	2.189	2.127	2.184	2.114	2.667	2.577	2.660	2.556	2.325	2.312	2.324	2.309
−2.0	1.521	1.518	1.521	1.518	1.783	1.743	1.779	1.735	2.056	1.974	2.048	1.958	1.780	1.777	1.779	1.777

Table 5 and Fig. 6 present finite differences results for uniform loading with two equal end moments. Values given by AISC LRFD [2] are non-conservative for $k = 0.5$ and $\beta < 1.3$ approximately, and are very conservative for $k = 1$ and $\beta > 1.1$, particularly if $k_w = 0.5$. Similar conclusions are found for uniform loading and one end moment. These results are given in Table 6 and Fig. 7. As expected, in both loading cases there is total agreement between Suryoatmono and Ho [25] and the values presented in this paper for $k = k_w = 1$.

Finally, Table 7 and Fig. 8 give the C_1 values for a concentrated load acting in the midspan and two equal end moments. The pattern is similar to previous cases, with AISC

Table 8
C_1 values for concentrated load and one end moment (Finite Differences)

	Finite Differences results															
	$k=1, k_w=1$				$k=1, k_w=0.5$				$k=0.5, k_w=1$				$k=0.5, k_w=0.5$			
	IPE500		HEB500		IPE500		HEB500		IPE500		HEB500		IPE500		HEB500	
M/M_0	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$	$L=8$	$L=16$
2.0	1.131	1.126	1.130	1.125	1.244	1.228	1.243	1.224	1.078	1.052	1.076	1.048	1.015	1.013	1.015	1.012
1.0	1.210	1.206	1.209	1.205	1.313	1.299	1.312	1.296	1.088	1.065	1.086	1.061	1.037	1.036	1.037	1.036
0.5	1.272	1.269	1.271	1.268	1.366	1.355	1.365	1.352	1.089	1.068	1.087	1.065	1.050	1.049	1.050	1.049
0.0	1.359	1.355	1.359	1.354	1.442	1.432	1.441	1.430	1.078	1.061	1.076	1.058	1.061	1.059	1.061	1.058
−0.5	1.471	1.463	1.471	1.462	1.546	1.536	1.546	1.534	1.031	1.020	1.030	1.019	1.055	1.050	1.055	1.049
−0.6	1.492	1.483	1.491	1.482	1.570	1.559	1.569	1.556	1.013	1.003	1.012	1.002	1.048	1.042	1.048	1.041
−0.7	1.623	1.614	1.622	1.612	1.714	1.701	1.713	1.699	1.066	1.057	1.065	1.055	1.117	1.110	1.117	1.108
−1.0	2.919	2.910	2.918	2.908	3.240	3.203	3.237	3.194	1.754	1.741	1.753	1.739	1.931	1.905	1.929	1.900
−1.5	3.901	3.651	3.870	3.617	5.147	4.729	5.106	4.651	2.919	2.829	2.908	2.816	3.373	3.220	3.359	3.190
−2.0	3.433	3.212	3.405	3.183	4.392	4.043	4.357	3.981	3.403	3.195	3.377	3.166	3.629	3.406	3.607	3.366

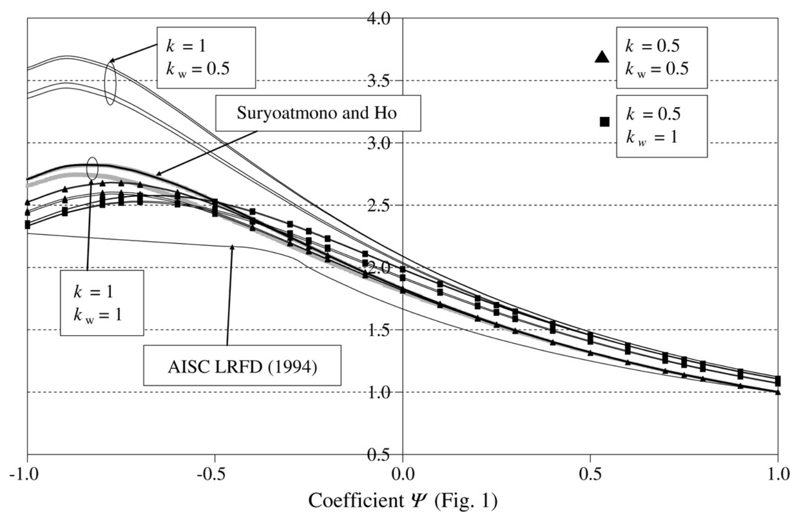

Fig. 5. Results for linear moment distribution (Table 4).

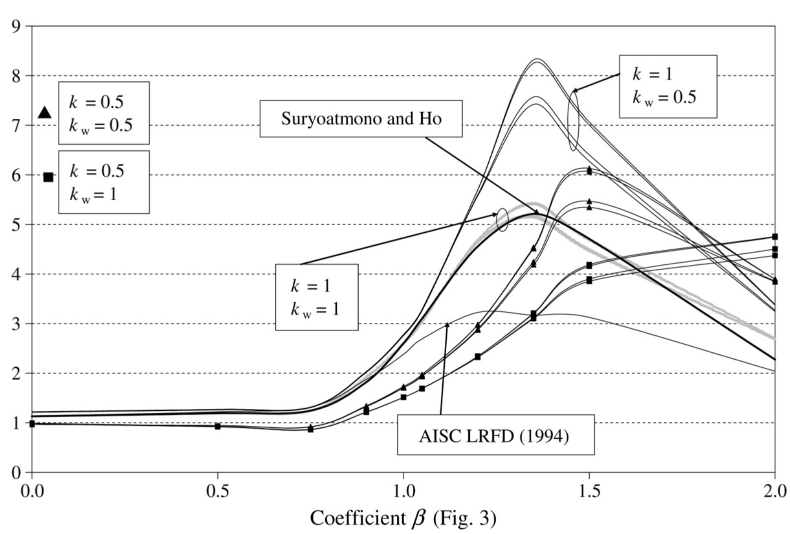

Fig. 6. Results for uniform loading and two end moments (Table 5).

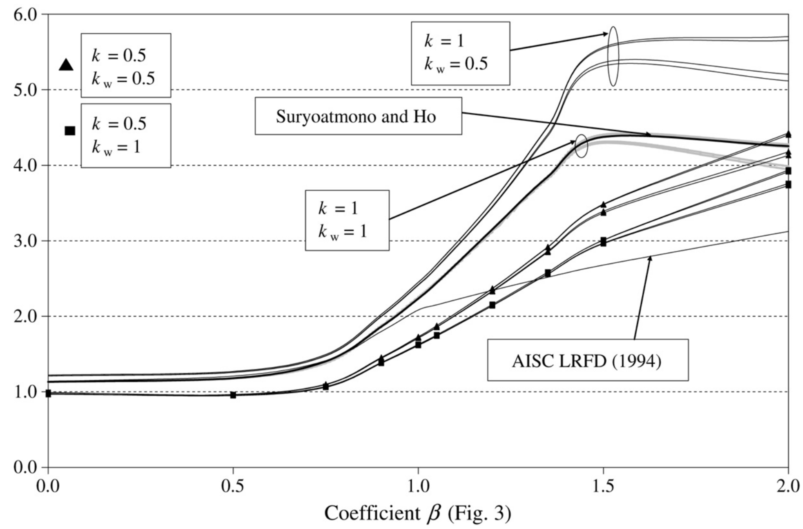

Fig. 7. Results for uniform loading and one end moment (Table 6).

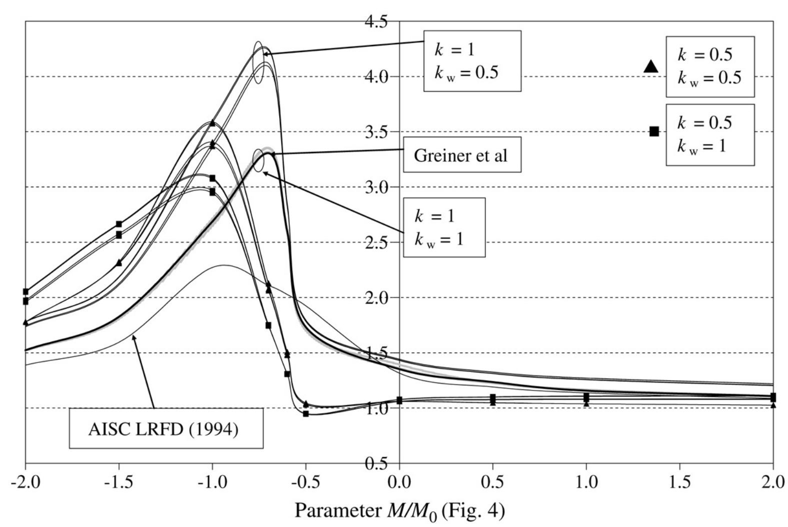

Fig. 8. Results for concentrated load and two end moments (Table 7).

LRFD [2] giving non-conservative values or very conservative values depending on the restriction to lateral bending and the M/M_0 quotient.

For the concentrated load and one end moment case, Table 8 and Fig. 9 show that the values given by AISC LRFD [2] are non-conservative for most values of M/M_0 quotient when $k = 0.5$.

The new C_1 values obtained using finite differences follow some general rules. First, restriction to lateral bending leads generally to lower values for the coefficient C_1. Second, prevention of warping significantly increases C_1 values. And third, errors obtained when using AISC LRFD [2] may be significant.

5. General closed-form expression for moment gradient factor

The use of a general closed-form expression for computing moment gradient factors, C_1, such as those proposed by AISC LRFD [2] and BS 5950-1 [5], has two major advantages. First, the designer may obtain the value of C_1 for any bending moment distribution without need of interpolation. Second, of great importance in a strongly computer-based design process, structural analysis software can directly incorporate the closed-form expression in the design check routines.

The closed-form expression proposed by AISC LRFD [2] and BS 5950-1 [5], given by Eqs. (4) and (5), provides very good approximations for most of the cases considered in this paper. However, these equations give conservative values when dealing with simply supported beams subject to some moment distribution and non-conservative values for cases with lateral bending and warping prevented.

Even though further research might provide more exact matching, a new formulation is proposed here for the equivalent uniform moment factor that gives a better approximation to numerical results than those given by Eqs. (4) and (5). Moreover, the new proposed formulation may be applied to situations where lateral bending and warping are prevented at end supports.

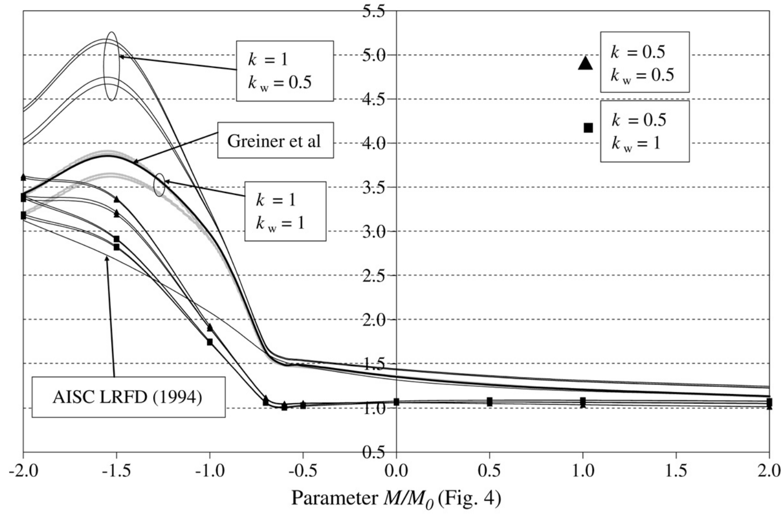

Fig. 9. Results for concentrated load and one end moment (Table 8).

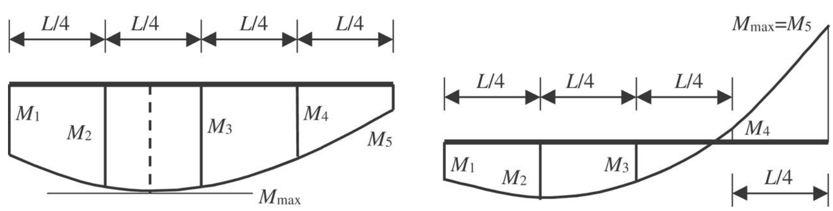

Fig. 10. Moment diagrams and moment values for Eqs. (14) and (15).

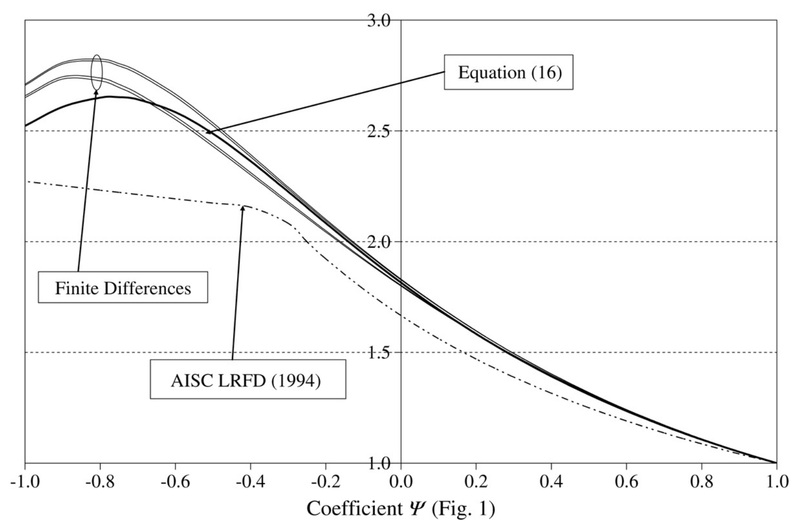

Fig. 11. Eq. (16) results for linear moment distribution ($k = 1$).

For a general moment diagram, the coefficient C_1 may be obtained by the following general closed expression:

$$C_1 = \frac{\sqrt{\sqrt{k}A_1 + [\frac{(1-\sqrt{k})}{2}A_2]^2} + \frac{(1-\sqrt{k})}{2}A_2}{A_1} \tag{13}$$

where

$$A_1 = \frac{M_{\max}^2 + 9kM_2^2 + 16M_3^2 + 9kM_4^2}{[1 + 9k + 16 + 9k]M_{\max}^2} \tag{14}$$

$$A_2 = \left|\frac{M_{\max} + 4M_1 + 8M_2 + 12M_3 + 8M_4 + 4M_5}{37M_{\max}}\right|. \tag{15}$$

Coefficient k is related to the lateral bending and warping prevention at end supports. It is equal to 1 if lateral bending and warping are free and equal to 0.5 if lateral bending and warping are prevented. Bending moments M_1 to M_5 are defined in Fig. 10. In Eqs. (14) and (15) each bending moment must have its corresponding sign.

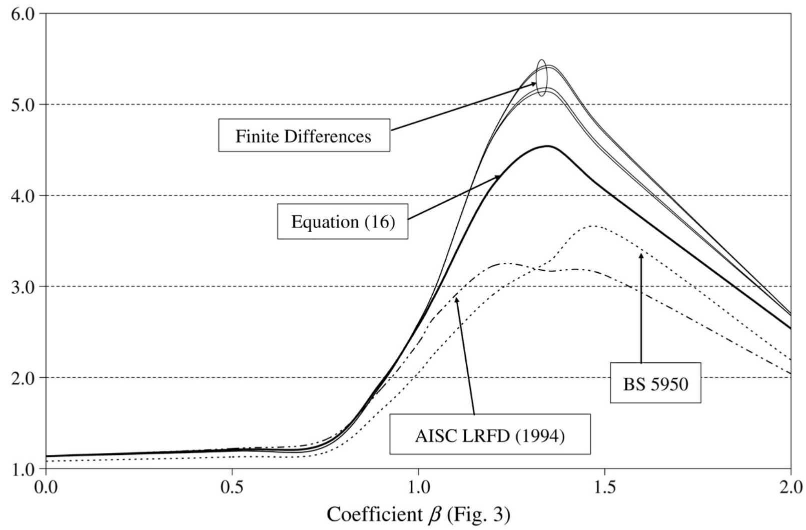

Fig. 12. Eq. (16) results for uniform loading and two end moments ($k = 1$).

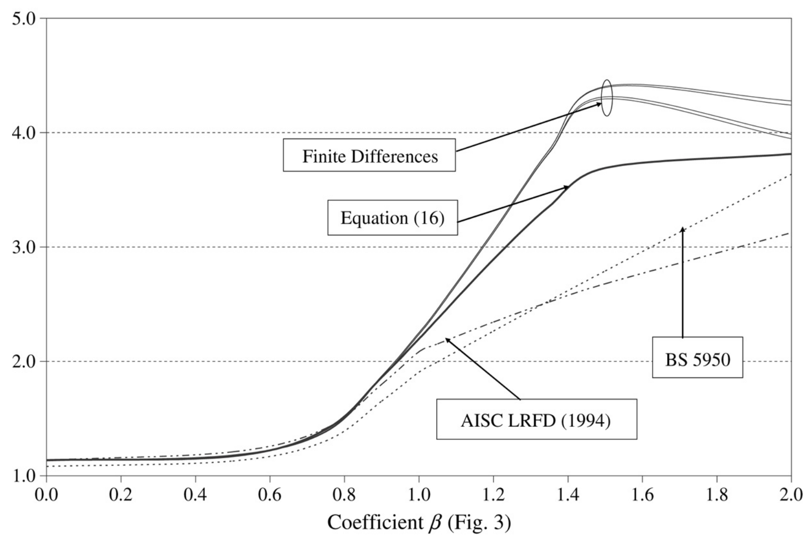

Fig. 13. Eq. (16) results for uniform loading and one end moment ($k = 1$).

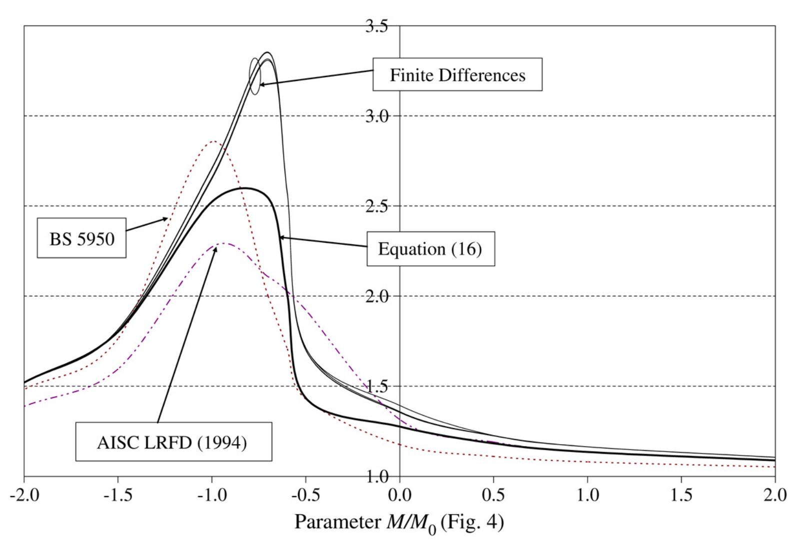

Fig. 14. Eq. (16) results for concentrated load and two end moments ($k = 1$).

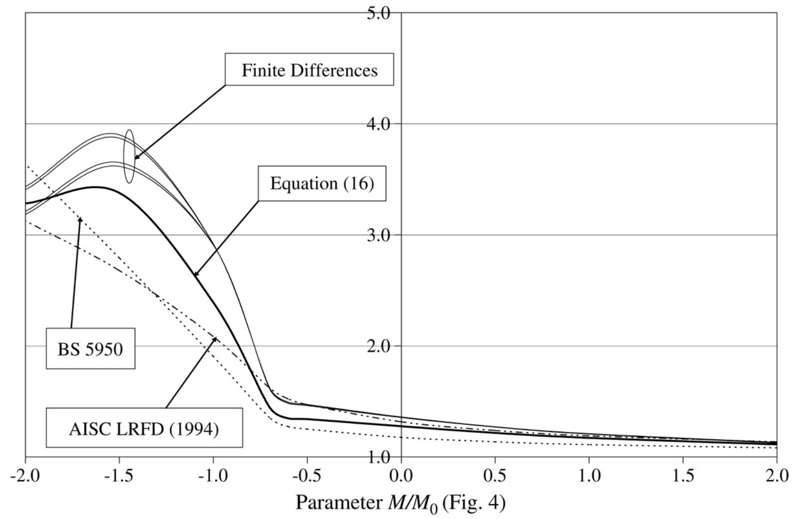

Fig. 15. Eq. (16) results for concentrated load and one end moment ($k = 1$).

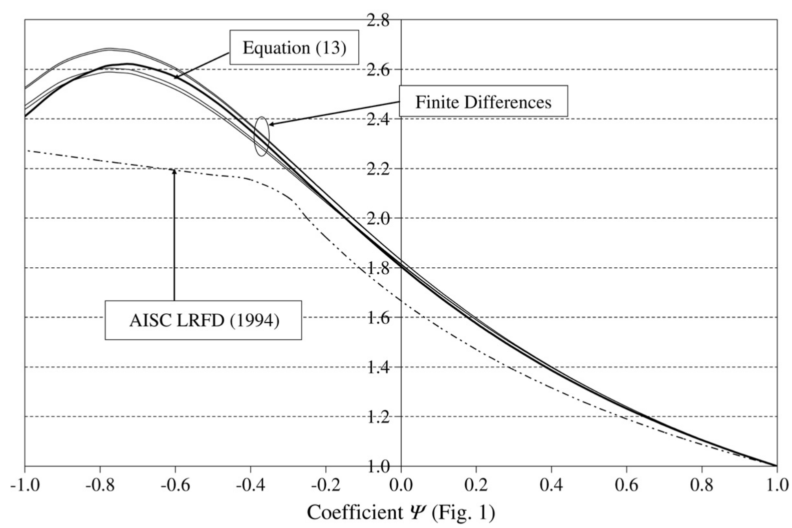

Fig. 16. Eq. (13) results for linear moment distribution ($k = 0.5$).

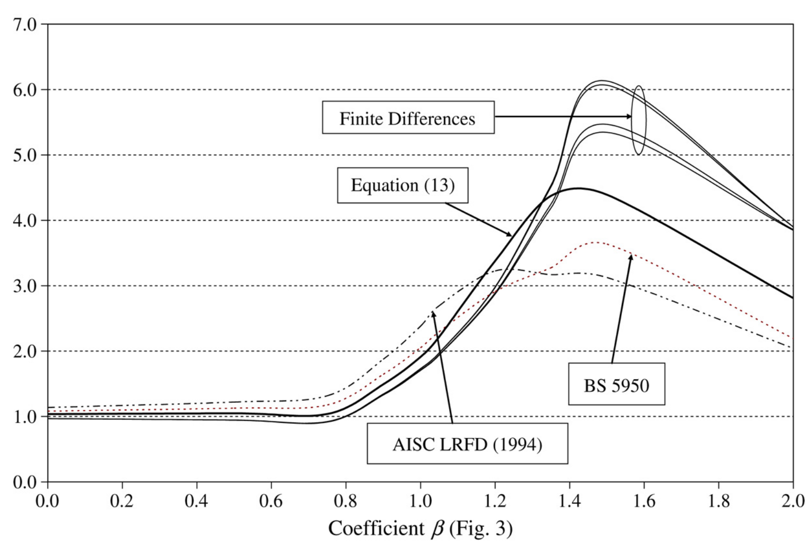

Fig. 17. Eq. (13) results for uniform loading and two end moments ($k = 0.5$).

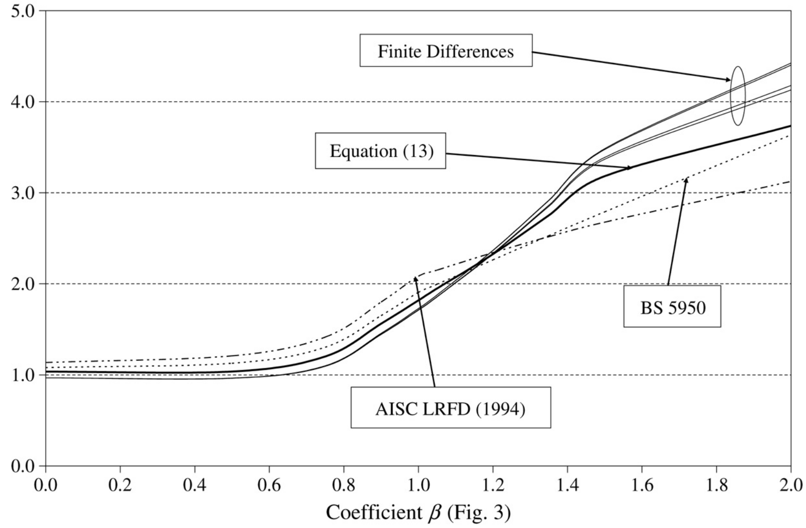

Fig. 18. Eq. (13) results for uniform loading and one end moment ($k = 0.5$).

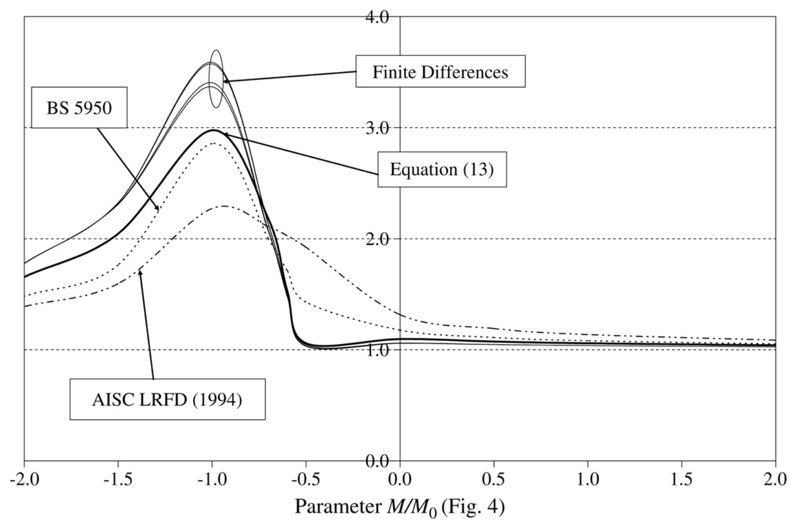

Fig. 19. Eq. (13) results for concentrated load and two end moments ($k = 0.5$).

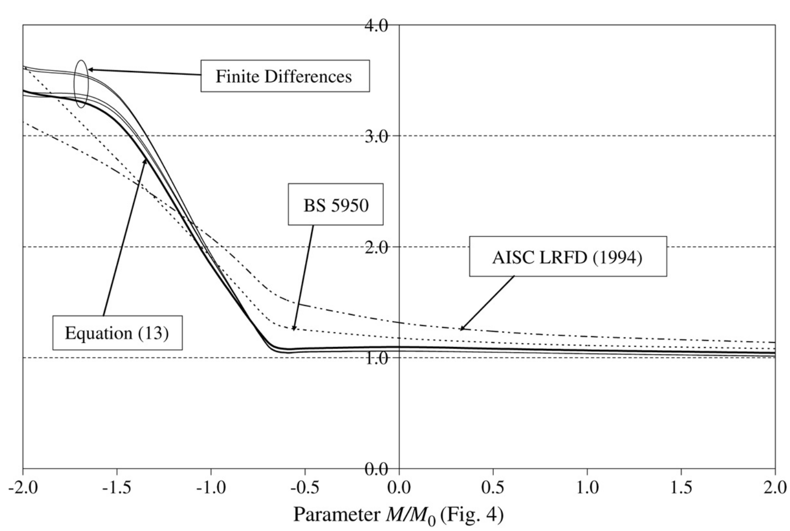

Fig. 20. Eq. (13) results for concentrated load and one end moment ($k = 0.5$).

For lateral bending and warping free at both end supports k is equal to 1 and Eq. (13) is simplified to

$$C_1 = \sqrt{\frac{35M_{\max}^2}{M_{\max}^2 + 9M_2^2 + 16M_3^2 + 9M_4^2}}. \qquad (16)$$

Figs. 11–15 compare the value of C_1 given by Eq. (16) and the numerical results and codes presented above for simply supported beams. Figs. 16–20 do the same for beams with lateral bending and warping prevented at supports, using for this case Eq. (13). It can be seen that in most cases the new closed-form equation proposed produces better results than those obtained using AISC LRFD [2] and BS 5950-1 [5].

6. Conclusions

As indicated in the Introduction, structural stability is a key point in the design of steel structures. Modern design standards when dealing with lateral–torsional buckling require the computation of the elastic critical moment, which greatly depends on the moment distribution along the beam and the conditions at end supports.

In this context, the paper has presented a review of equivalent uniform moment factors (C_1) for lateral torsional buckling of steel beams proposed in the literature and a summary of design codes procedures. The review shows that enough information is available for beams with no prevention to both lateral bending and warping at end supports. For these cases, there is quite good agreement between the values given by design codes and the numerical results obtained using finite elements and finite differences approaches. In contrast, very few numerical results can be found for cases with lateral bending prevented and warping prevented. Moreover, for these few cases available, design codes values are sometimes very conservative and sometimes non-conservative.

As one of its main contributions, the paper has presented a set of new results obtained using both finite elements and finite differences. The new values refer to the equivalent uniform moment factor for the following cases: linear moment distributions; uniform distributed loading, with two and one end moments; and concentrated load, with two and one end moments. All these cases have been solved considering all possible end support conditions: no prevention to lateral bending and warping; prevention to lateral bending and warping; and prevention to lateral bending or prevention to warping. The results show that warping prevention leads to a significant increase in the coefficient C_1. Beams with lateral bending prevention at end supports have lower values of C_1 than those free to bend. The results for the new cases confirm that values given by design codes, besides being too conservative in many cases, do not properly approximate cases with prevention to lateral bending and warping, which might well be non-conservative.

Finally, the paper has presented a closed-form expression to obtain the equivalent uniform moment factor C_1 for any moment distribution. In most cases, the new expression renders values that are significantly closer to numerical results than those provided by the similar expression contained in AISC LRFD and BS 5950. The proposed closed-form formula incorporates the end support conditions through a parameter related to the lateral torsional buckling length of the beam.

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