The stiffness of plates

1. Introduction

The word **'plate'** is a collective term for elements in which forces can be transferred in two directions. Floors, walls, bridge slabs and laminates are all plates.

Hollow core floor | Concrete walls | Ribbed floor | Laminate

Figure 1: Examples of plates

Plates can be loaded in their plane and perpendicular to their plane. The **membrane behaviour** describes how a plate reacts to loading in its plane. The **bending behaviour**[1] describes how a plate reacts to loading perpendicular to its plane.

To describe the behaviour we will need **the rigidity** of the plate. Rigidity is a measure for the resistance of an element against deformation (thus strain ε and curvatures κ).

Once the rigidity is known, Diamonds will use it in the **Finite Element Method**[2]. The FEM chops the structure into a finite number of elements, which will then be logically linked to each other, allowing Diamonds to calculate the displacements of the structure. The stresses (and resulting sectional forces) can then be derived from the displacements because of these relations between them:

- The kinematic equations give the relations between the displacements and the deformations.
- The constitutive equations give information on the material behaviour, by providing the relations between the stresses and the strains.
- The equilibrium equations give the relations between the loads and the stresses.

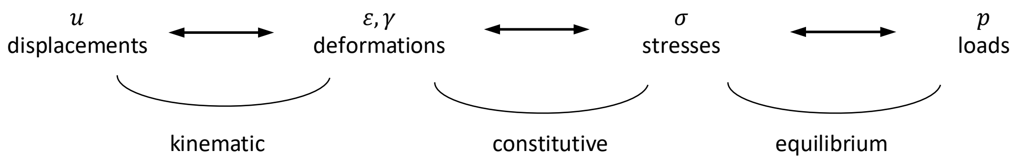


Figure 2: Scheme of relationships in a plate

With the aim of determining the rigidity of plates, the focus of this document lies on the constitutive equation. More information on the other relations can be found in for example [2].

[1] The shear behaviour is not handled in this document since only Thin Plate Theory (neglecting the shear deformation of the plate) is implemented in Diamonds. The Thick Plate Theory (taking the shear deformation for plates into account) is not implemented. Thin Plate Theory is also called 'Kirchoff Theory', Thick Plate Theory is also called 'Mindlin Theory'.

[2] short 'FEM', synonym 'displacement method'

2. An isotropic plate

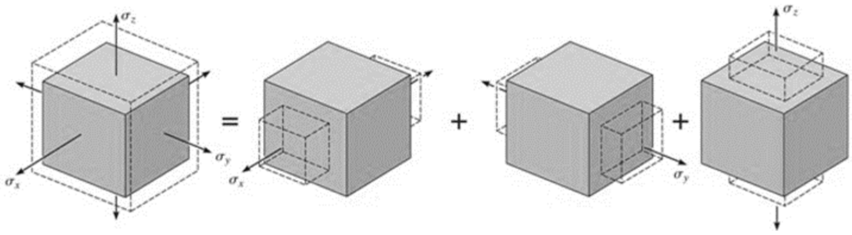
The simplest form of a plate is an **isotropic** plate. The word ‘isotropic’ refers to the material behaviour and means ‘homogeneous in all directions’, like solid steel. §2.1 describes the stress-strain relation for an isotropic material so the sectional forces (membrane forces and bending moments) can be calculated in §2.2. The example in §2.3 shows how the formula are used.

2.1. Constitutive equations

2.1.1. Hooke’s law in 3D

If the material is subjected to a state of triaxial stress, associated normal strains will be developed in the material. The total strain in a direction equals the sum of all strains in that direction due to the stresses in each direction [4]:

$$\begin{gathered}\varepsilon_x = \varepsilon_{x\ due\ to\ \sigma\_x} + \varepsilon_{x\ due\ to\ \sigma\_y} + \varepsilon_{x\ due\ to\ \sigma\_z} \\ \varepsilon_y = \varepsilon_{y\ due\ to\ \sigma\_x} + \varepsilon_{y\ due\ to\ \sigma\_y} + \varepsilon_{y\ due\ to\ \sigma\_z} \\ \varepsilon_z = \varepsilon_{x\ due\ to\ \sigma\_x} + \varepsilon_{z\ due\ to\ \sigma\_y} + \varepsilon_{z\ due\ to\ \sigma\_z}\end{gathered} \tag{1}$$



Taking Hooke’s law ($\sigma = E \cdot \varepsilon$) and Poisson’s ratio υ into account, the expressions (1) transform into:

$$\begin{gathered}\varepsilon_x = \varepsilon_{x\ due\ to\ \sigma\_x} + \varepsilon_{x\ due\ to\ \sigma\_y} + \varepsilon_{x\ due\ to\ \sigma\_z} \\ \varepsilon_x = \frac{\sigma_x}{E} - \upsilon\frac{\sigma_y}{E} - \upsilon\frac{\sigma_z}{E} \\ \varepsilon_x = \frac{1}{E}\left(\sigma_x - \upsilon\sigma_y - \upsilon\sigma_z\right)\end{gathered} \tag{2}$$

$$\begin{gathered}\varepsilon_y = \varepsilon_{y\ due\ to\ \sigma\_x} + \varepsilon_{y\ due\ to\ \sigma\_y} + \varepsilon_{y\ due\ to\ \sigma\_z} \\ \varepsilon_y = \frac{1}{E}\left(\sigma_y - \upsilon\sigma_x - \upsilon\sigma_z\right); \varepsilon_z\end{gathered} \tag{3}$$

$$\begin{gathered}\varepsilon_z = \varepsilon_{x\ due\ to\ \sigma\_x} + \varepsilon_{z\ due\ to\ \sigma\_y} + \varepsilon_{z\ due\ to\ \sigma\_z} \\ \varepsilon_z = \frac{1}{E}\left(\sigma_z - \upsilon\sigma_x - \upsilon\sigma_y\right)\end{gathered} \tag{4}$$

If we apply a shear stress τ_{xy} to the element, the material will deform only due to a shear strain γ_{xy}; that is τ_{xy} will not cause other strains in the material. Likewise τ_{yz} and τ_{xz} will only cause shear strains γ_{yz} and γ_{xz}.

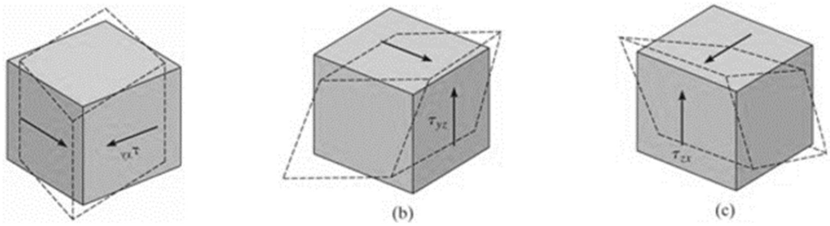


Figure 3: Hooke's law for shear stress

Hooke's law relating shear stress and shear strain is:

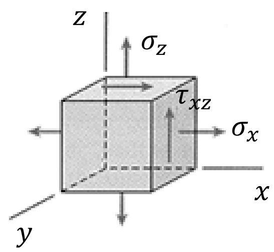
$$\gamma_{xy} = \frac{1}{G}\tau_{xy};\ \gamma_{yz} = \frac{1}{G}\tau_{yz};\ \gamma_{xz} = \frac{1}{G}\tau_{xz} \tag{5}$$

Write equations (2), (3), (4) and (5) in matrix form (stress-strain) and Hooke's law for a linear elastic material in 3D is obtained:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \frac{E(1-v)}{(1-2v)(1+v)} & \frac{E \cdot v}{(1-2v)(1+v)} & \frac{E \cdot v}{(1-2v)(1+v)} & 0 & 0 & 0 \\ \frac{E \cdot v}{(1-2v)(1+v)} & \frac{E(1-v)}{(1-2v)(1+v)} & \frac{E \cdot v}{(1-2v)(1+v)} & 0 & 0 & 0 \\ \frac{E \cdot v}{(1-2v)(1+v)} & \frac{E \cdot v}{(1-2v)(1+v)} & \frac{E(1-v)}{(1-2v)(1+v)} & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & sym & & 0 & G & 0 \\ & & & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{6}$$

2.1.2. Hooke's law in 2D

If the plate is thin and there are no out-of-plane loads, it can be considered to be under **plane stress**. Than $\sigma_y = 0$, $\tau_{yz} = \tau_{xy} = 0$. Equation (6) simplifies to:



$$\begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{bmatrix} = \underbrace{\frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}}_{[Q]} \begin{bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{bmatrix} \tag{7}$$

The 3x3 matrix $[Q]$ in equation (7) is called the **elasticity matrix**.

Note: an alternative way to transform Hooke's law from 3D to 2D is plane strain. But since Diamonds assumes plane stress, this is not treated.

2.2. Sectional forces

The resultant membrane forces and moments can be calculated as the sum of the stresses (equ. (7)) over the thickness t of the plate:

$$\begin{Bmatrix} N_{xx} \\ N_{zz} \\ N_{xz} \end{Bmatrix} = \int_{-\frac{t}{2}}^{-\frac{t}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} dy$$

$$\begin{Bmatrix} M_{xx} \\ M_{zz} \\ M_{xz} \end{Bmatrix} = \int_{-\frac{t}{2}}^{-\frac{t}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} y\, dy$$

This results in the following matrix equations (with nor σ_{xx} nor ε depended of the thickness):

$$\begin{Bmatrix} N_{xx} \\ N_{zz} \\ N_{xz} \end{Bmatrix} = t \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix} = t \cdot [Q] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix} = \underbrace{\begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix}}_{[d]} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} M_{xx} \\ M_{zz} \\ M_{xz} \end{Bmatrix} = \frac{t^3}{12} \frac{E}{(1-\upsilon^2)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{zz} \\ \rho_{xz} \end{Bmatrix} = \frac{t^3}{12} \cdot [Q] \begin{Bmatrix} \kappa_{xx} \\ \kappa_{zz} \\ \rho_{xz} \end{Bmatrix} = \underbrace{\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}}_{[D]} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{zz} \\ \rho_{xz} \end{Bmatrix} \tag{9}$$

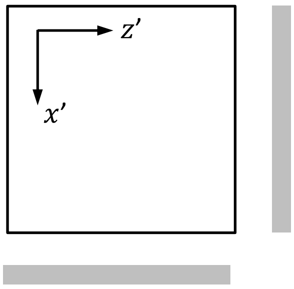
Equation (8) expresses the membrane behaviour of an isotropic plate, equation (9) the bending behaviour.

Notes:

- The 3x3 matrix $[d]$ in equation (8) is called the **plate membrane stiffness matrix.** The 3x3 matrix $[D]$ in equation (9) is called the **plate bending stiffness matrix.**
- In Diamonds equations (8) and (9) are combined in one matrix equation. And $d_{16} = d_{26} = D_{16} = D_{26} = 0$ (more info in §**Fout! Verwijzingsbron niet gevonden.** :

$$\begin{Bmatrix} N_{xx} \\ N_{zz} \\ N_{xz} \\ M_{xx} \\ M_{zz} \\ M_{xz} \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & 0 & 0 & 0 & 0 \\ d_{12} & d_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{66} & 0 & 0 & 0 \\ & & & D_{11} & D_{12} & 0 \\ & sym & & D_{12} & D_{22} & 0 \\ & & & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \\ \kappa_{xx} \\ \kappa_{zz} \\ \rho_{xz} \end{Bmatrix} \tag{10}$$

2.3. Example: isotropic geometry – isotropic material



$$t = 200mm$$

Concrete C25/30:

$$E = 31\,476N/mm^2, \upsilon = 0{,}2$$

2.3.1. Manual calculation of the stiffness properties

We use equation (8) to describe the membrane behaviour.

$$d_{11} = d_{22} = \frac{E \cdot t}{1 - \upsilon^2} = \frac{31\,476N/mm^2 \cdot 200mm}{1 - 0{,}2^2} = 6\,557 \cdot 10^3 kN/m$$

$$d_{12} = d_{21} = \upsilon \frac{E \cdot t}{1 - \upsilon^2} = \upsilon \cdot d_{11} = 0{,}2 \cdot 6557 \cdot 10^3 kN/m = 1\,311 \cdot 10^3 kN/m$$

$$d_{66} = \frac{1}{2}(1 - \upsilon)\frac{E \cdot t}{1 - \upsilon^2} = \frac{1}{2}(1 - \upsilon) \cdot d_{11} = \frac{1}{2}(1 - 0{,}2) \cdot 6557 \cdot 10^3 kN/m = 2\,623 \cdot 10^3 kN/m$$

We use equation (9) to describe the bending behaviour.

$$D_{11} = D_{22} = \frac{E \cdot t^3}{12(1 - \upsilon^2)} = \frac{31\,476N/mm^2 \cdot (200mm)^3}{12(1 - 0{,}2^2)} = 21\,858kNm$$

$$D_{12} = D_{21} = \upsilon \frac{E \cdot t^3}{12(1 - \upsilon^2)} = \upsilon \cdot D_{11} = 0{,}2 \cdot 21\,858kNm = 4\,372kNm$$

$$D_{66} = \frac{1}{2}(1 - \upsilon)\frac{E \cdot t^3}{12(1 - \upsilon^2)} = \frac{1}{2}(1 - \upsilon) \cdot D_{11} = \frac{1}{2}(1 - 0{,}2) \cdot 21\,858kNm = 8\,743kNm$$

2.3.2. Comparison with Diamonds

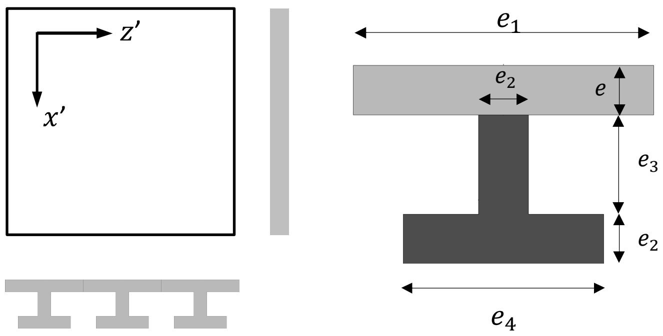
The stiffness matrix of an isotropic plate in Diamonds gives the same results as calculated by hand:

Note: If you want to compare the stiffness matrix in Diamonds to manual calculations, make sure the correct standard (here EN 1992-1-1 [--]) is selected. Some materials have a different Young's modulus depending on the standard/ national annex.

3. Other plate types

Many plate types cannot be handled as an isotropic plate. Stiffeners may occur and they can be different in two orthogonal directions (§3.1). Not all materials have isotropic properties, a material can have different properties in two mutually perpendicular directions. This type of material is called **orthotropic** (§3.2). A plate can also be composed of multiple orthotropic layers. This type of plate is called a **laminate** (§3.3).

3.1. Orthotropic geometry (single I-slab) – isotropic material



$e = 100mm, e_1 = 600mm, e_2 = 100mm$
$e_3 = 200mm, e_4 = 400mm$

$A_a = 50\,000\ mm^2$ (dark grey zone)
$I_{xx} = 2\,325\,000\,000\ mm^4$ (moment of inertia light + dark grey zone)

Concrete C25/30:
$E = 31\,476 N/mm^2$
$v = 0{,}2$

In this example we will replace the real shape of the plate with a solid plate taking the geometry of the stiffeners into account. This method is referred to as **shape-orthotropy**. It can be used for plates with **repeating** stiffeners with **regular spacing**.

The formula for the stiffness components are [2]:

$$[d] = \begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix} = \begin{bmatrix} \underbrace{\frac{E \cdot t}{1 - v^2}}_{\substack{rigidity\ of \\ isotropic \\ plate}} + \underbrace{\frac{E \cdot A_a}{a}}_{\substack{rigidity\ of \\ I-shaped\ parts \\ smeared\ out}} & v \cdot d_{22} & 0 \\ v \cdot d_{22} & \frac{E \cdot t}{1 - v^2} & 0 \\ 0 & 0 & \frac{1}{2}(1 - v) \cdot d_{22} \end{bmatrix}$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66.av} \end{bmatrix} = \begin{bmatrix} \frac{E \cdot I_{xx}}{a} & v \cdot D_{22} & 0 \\ v \cdot D_{22} & \frac{E \cdot t^3}{12(1 - v^2)} & 0 \\ 0 & 0 & 0{,}5 \cdot G \cdot \frac{i_{12} + i_{21}}{2} \end{bmatrix}$$

The torsional rigidity D_{66} needs special attention. Using here the formula for an isotropic plate will underestimate the torsional rigidity, because the stiffeners are neglected. The alternative is to calculate the torsional rigidities per units length in each direction (i_{xz} and i_{zx}) and take the average, resulting in $D_{66.av}$. Formula for the rigidity per units length are given in Figure 4: Torsional rigidity per unit length.

	TORSIONAL RIGIDITY PER UNIT LENGTH $\boldsymbol{i}$
ISOTROPIC PLATE	$\frac{e^3}{6}$
RECTANGLE ($\boldsymbol{b} > \boldsymbol{a}$)	$i(a,b) = \frac{\frac{a^3 b}{16}\left[\frac{16}{3} + 3{,}36\frac{a}{b}\left(1 - \frac{a^4}{12b^4}\right)\right]}{e_1}$

Figure 4: Torsional rigidity per unit length [2], [7]

3.1.1. Manual calculation of the stiffness properties

$$d_{11} = \frac{31\,476N/mm^2 \cdot 100mm}{(1-0{,}2^2)} + \frac{31\,476N/mm^2 \cdot 50\,000mm^2}{600mm} = 5\,902\ \cdot 10^3 kN/m$$
$$d_{22} = \frac{31\,476N/mm^2 \cdot 100mm}{(1-0{,}2^2)} = 3\,279\ \cdot 10^3 kN/m$$
$$d_{12} = d_{21} = 656\ \cdot 10^3 kN/m$$
$$d_{66} = \frac{1}{2}(1-0{,}2) \cdot 3\,279\ \cdot 10^3 kN/m = 1\,312\ \cdot 10^3 kN/m$$

$$D_{11} = \frac{31\,476N/mm^2 \cdot 2\,325\,000\,000\ mm^4}{600mm} = 121\,970kNm$$
$$D_{22} = \frac{31\,476N/mm^2 \cdot (100mm)^3}{12(1-0{,}2^2)} = 2\,732\ kNm$$
$$D_{12} = D_{21} = 0{,}2 \cdot D_{22} = 0.2 \cdot 2\,732kNm = 546\ kNm$$

To calculate the torsional rigidity in the z-direction i_{12}, the I-section can be seen as a sum of rectangles [7]. In the x'-direction i_{21} we use the formula for an isotropic plate.

$$i_{12} = \frac{i(e,e_1) + i(e_2, e_3 - e_4) + i(e_4, e_5)}{e_1} = \frac{i(100mm, 600mm) + i(100mm, 200mm) + i(100mm, 200mm)}{600mm}$$
$$= 771\,470mm^3$$

$$i_{21} = \frac{e^3}{6} = \frac{(100mm)^3}{6} = 166\,667mm^3$$

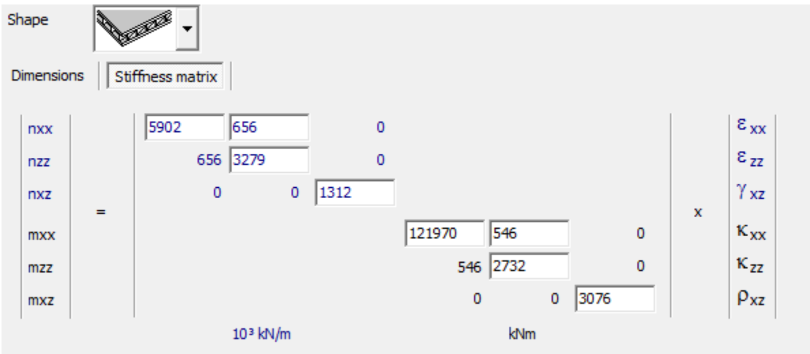
$$D_{66.av} = 0{,}5 \cdot \frac{31\,476N/mm^2}{2(1+0{,}2)} \cdot \frac{(771\,470 + 166\,667)mm^3}{2} = 3076kNm$$

3.1.2. Enter the stiffness in Diamonds

To use the calculated stiffness in Diamonds:

- Select this shape 

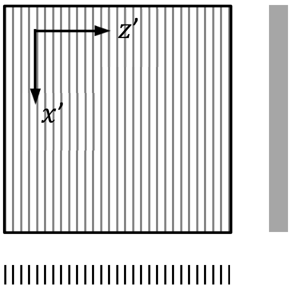
.
- In the tab page *Dimensions*: assign a thickness and a material to the plate. Diamonds will use this to calculate the self-weight of the plate.
- Go to the tab page *Stiffness matrix*. Enter the values for the stiffness.



Notes:

- The stiffness matrix was determined based on the real shape of the plate, but the real shape itself is unknown to Diamonds. Therefor it is not possible to calculate elastic stresses, reinforcement or creeped deformation for this plate type.
  If you do want to take those effects into account, they should be implemented in the stiffness matrix.

3.2. Isotropic geometry – orthotropic material



$$t = 20mm$$

Norway Spruce (Dinwoodie, 2000):

$$E_1 = 10\,700N/mm^2\ (x' - direction)$$
$$E_2 = 430N/mm^2\ (z' - direction)$$
$$G = 620N/mm^2$$
$$\upsilon_{12} = 0.51, \upsilon_{21} = 0.02$$

The formula for the stiffness components are [5]:

$$[d] = \begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix} = \begin{bmatrix} \dfrac{E_1 \cdot t}{1 - \upsilon_{12}\upsilon_{21}} & \upsilon_{21} \cdot d_{11} = \upsilon_{12} \cdot d_{22} & 0 \\ \upsilon_{21} \cdot d_{11} = \upsilon_{12} \cdot d_{22} & \dfrac{E_2 \cdot t}{1 - \upsilon_{12}\upsilon_{21}} & 0 \\ 0 & 0 & G \cdot t \end{bmatrix}$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66.av} \end{bmatrix} = \begin{bmatrix} \dfrac{E_1 \cdot t^3}{12(1 - \upsilon_{12}\upsilon_{21})} & \upsilon_{21} \cdot D_{11} = \upsilon_{12} \cdot D_{22} & 0 \\ \upsilon_{21} \cdot D_{11} = \upsilon_{12} \cdot D_{22} & \dfrac{E_2 \cdot t^3}{12(1 - \upsilon_{12}\upsilon_{21})} & 0 \\ 0 & 0 & 0{,}5 \cdot G \cdot \dfrac{i_{12} + i_{21}}{2} \end{bmatrix}$$

We recognize these formula for the elasticity matrix $[Q]$:

$$Q_{11} = \frac{E_1}{1 - \upsilon_{12}\upsilon_{21}};\ Q_{22} = \frac{E_2}{1 - \upsilon_{12}\upsilon_{21}}; Q_{12} = Q_{21} = \upsilon_{21} \cdot Q_{11} = \upsilon_{12} \cdot Q_{22}; Q_{66} = G \qquad (11)$$

3.2.1. Manual calculation of the stiffness properties

$$d_{11} = \frac{10\ 700N/mm^2 \cdot 20mm}{1 - 0.02 \cdot 0.51} = 216.2\ \cdot 10^3 kN/m$$

$$d_{22} = \frac{430N/mm^2 \cdot 20mm}{1 - 0.02 \cdot 0.51} = 8.7\ \cdot 10^3 kN/m$$

$$d_{12} = d_{21} = 0.02 \cdot 216.2\ \cdot 10^3 kN/m = 4.3 \cdot 10^3 kN/m$$

$$d_{66} = 620N/mm^2 \cdot 20mm = 12.4 \cdot 10^3 kN/m$$

$$D_{11} = \frac{10\ 700N/mm^2 \cdot (20mm)^3}{12(1 - 0.02 \cdot 0.51)} = 7.21kNm$$

$$D_{22} = \frac{370N/mm^2 \cdot (20mm)^3}{12(1 - 0.02 \cdot 0.51)} = 0.29kNm$$

$$D_{12} = D_{21} = 0.057 \cdot 7.21kNm = 0.14kNm$$

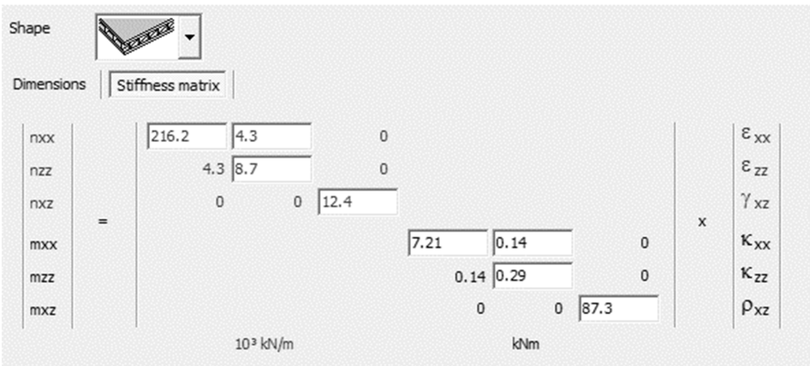
$$D_{66.av} = 0{,}5 \cdot 620N/mm^2 \cdot 0.5 \cdot \left(\frac{(20mm)^3}{6} + \frac{(20mm)^3}{6}\right) = 87.3kNm$$

3.2.2. Enter the stiffness in Diamonds

To use the calculated stiffness in Diamonds:

- Select this shape 

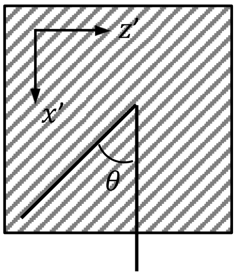
.
- In the tab page *Dimensions*: assign a thickness and a material to the plate. Diamonds will use this to calculate the self-weight of the plate.
- Go to the tab page *Stiffness matrix*. Enter the values for the stiffness.



Notes

- The stiffness matrix was determined based on the real shape of the plate, but the real shape itself is unknown to Diamonds. Therefor it is not possible to calculate elastic stresses, reinforcement or creeped deformation for this plate type.
  If you do want to take those effects into account, they should be implemented in the stiffness matrix.

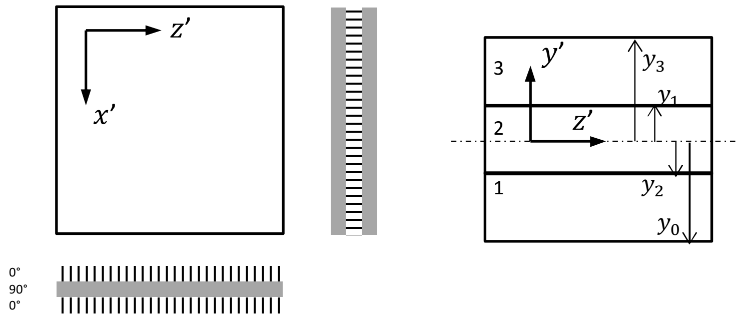
- If the principal axes of orthotropic are rotated over an angle θ in relation to the local x'z'-coordinate system, the elasticity matrix $[Q]$ will undergo a transformation to $[Q]_\theta$. In the transformed elasticity matrix $[Q]_\theta$ the components $Q_{16.\theta}$ and $Q_{26.\theta}$ are no longer equal to zero [6]. Diamonds assumes $Q_{16} = Q_{26} = 0$, thus a plate with rotated principal axes of orthotropic is not possible with Diamonds.



$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$

$$[Q]_\theta = \begin{bmatrix} Q_{11.\theta} & Q_{12.\theta} & Q_{16.\theta} \\ Q_{12.\theta} & Q_{22.\theta} & Q_{26.\theta} \\ Q_{16.\theta} & Q_{26.\theta} & Q_{66.\theta} \end{bmatrix}$$

3.3. Orthotropic geometry (laminate) – orthotropic material



3-ply plate

- each ply has a thickness of 20mm
- the plies are numbered from 1 to 3 starting from the bottom

Norway Spruce (Dinwoodie, 2000):

$$E_1 = 10\,700 N/mm^2\ (x' - direction)$$
$$E_2 = 430 N/mm^2\ (z' - direction)$$
$$G = 620 N/mm^2$$
$$\upsilon_{12} = 0.51, \upsilon_{21} = 0.02$$

For a multi-ply laminate the relation for the sectional forces is a bit more complex than in §2.2. The purpose is to take the sum of the contributions of all laminate plies k [1] (Kubiak, 2013).

$$\begin{Bmatrix} N_{xx} \\ N_{zz} \\ N_{xz} \end{Bmatrix} = \int_{t_{k-1}}^{t_k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} dy = \underbrace{\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} dy}_{\substack{sum\ of\ the\ contribution \\ of\ the\ various\ layers}}$$

$$\begin{Bmatrix} M_{xx} \\ M_{zz} \\ M_{xz} \end{Bmatrix} = \int_{t_{k-1}}^{t_k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} y\, dy = \underbrace{\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} y\, dy}_{\substack{sum\ of\ the\ contribution \\ of\ the\ various\ layers}}$$

Resulting in these matrix equations:

$$\begin{Bmatrix} N_{xx} \\ N_{zz} \\ N_{xz} \end{Bmatrix} = \underbrace{\begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix}}_{[d]} \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{zz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} + \underbrace{\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}}_{[B]} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{zz} \\ \rho_{xz} \end{Bmatrix} \tag{12}$$

$$\begin{Bmatrix} M_{xx} \\ M_{zz} \\ M_{xz} \end{Bmatrix} = \underbrace{\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}}_{[B]} \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{zz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} + \underbrace{\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}}_{[D]} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{zz} \\ \rho_{xz} \end{Bmatrix} \tag{13}$$

The components in the matrix $[d]$, $[B]$ and $[D]$ are calculated with the formula:

$$d_{ij} = \sum_{k-1}^{n} \left(Q_{ij}\right)_k (y_k - y_{k-1}) \tag{14}$$

$$B_{ij} = \frac{1}{2} \sum_{k-1}^{n} \left(Q_{ij}\right)_k (y_k^2 - y_{k-1}^2) \tag{15}$$

$$D_{ij} = \frac{1}{3} \sum_{k-1}^{n} \left(Q_{ij}\right)_k (y_k^3 - y_{k-1}^3) \tag{16}$$

The 3x3 matrix $[d]$ and $[D]$ in equations (12) and (13) are again the membrane and bending stiffness matrix. In literature the matrix [d] is referred to as matrix $[A]$. The 3x3 matrix $[B]$ in equations (12) and

(13) is called the **coupling matrix** and relates the in-plane forces and the out-of-plane deformations. Q_{ij} in equations (14), (15) and (16) is the relevant component of the elasticity matrix. y is the distance from the outer edge of a ply towards the mid-plane.

3.3.1. Manual calculation of the stiffness properties

The elasticity matrix for a 0° ply is (11):

$$Q_{11} = \frac{E_1}{1 - \upsilon_{12}\upsilon_{21}} = \frac{10\,700N/mm^2}{1 - 0.51 \cdot 0.02} = 10\,810N/mm^2$$

$$Q_{22} = \frac{E_2}{1 - \upsilon_{12}\upsilon_{21}} = \frac{430N/mm^2}{1 - 0.51 \cdot 0.02} = 434N/mm^2$$

$$Q_{12} = \upsilon_{21} \cdot Q_{11} = 0.02 \cdot 12\,163N/mm^2 = 222N/mm^2$$

$$Q_{66} = G = 620N/mm^2$$

$$Q_{16} = Q_{26} = 0$$

$$[Q]_{0°} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} 10\,810 & 222 & 0 \\ 222 & 434 & 0 \\ 0 & 0 & 620 \end{bmatrix}$$

In the elasticity matrix for a 90° ply, Q_{11} and Q_{22} switched places since the mean direction is rotated 90°:

$$[Q]_{90°} = \begin{bmatrix} \boldsymbol{Q_{22}} & Q_{12} & Q_{16} \\ Q_{12} & \boldsymbol{Q_{11}} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} 434 & 222 & 0 \\ 222 & 10\,810 & 0 \\ 0 & 0 & 620 \end{bmatrix}$$

The first component d_{11} of the membrane stiffness is calculated with equation (14):

$$\begin{aligned} d_{11} &= Q_{1.11}(30mm - 10mm) + Q_{2.11}\big(10mm - (-10mm)\big) + Q_{3.11}\big(-10mm - (-30mm)\big) \\ d_{11} &= 10\,810 \cdot (30mm - 10mm) + 434\big(10mm - (-10mm)\big) + 10\,810\big(-10mm - (-30mm)\big) \\ &= 441.1 \cdot 10^3 kN/m \end{aligned}$$

The other components can be found the same way:

$$d_{22} = 233.6 \cdot 10^3 kN/m$$

$$d_{12} = 13.3 \cdot 10^3 kN/m$$

$$d_{66} = 37.2 \cdot 10^3 kN/m$$

The first component D_{11} of the membrane stiffness is calculated with equation (16):

$$D_{11} = \frac{1}{3}\left[Q_{1.11}(30^3 - 10^3) + Q_{2.11}(10^3 - (-10)^3) + Q_{3.11}\big((-10)^3 - (-30)^3\big)\right]$$

$$D_{11} = \frac{1}{3}\left[10\,810(30^3 - 10^3) + 434(10^3 - (-10)^3) + 10\,810\big((-10)^3 - (-30)^3\big)\right] = 187.7kNm$$

The other components can be found the same way:

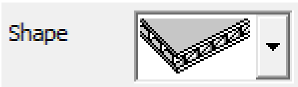
$$D_{22} = 14.7kNm$$

$$D_{12} = 4.0kNm$$

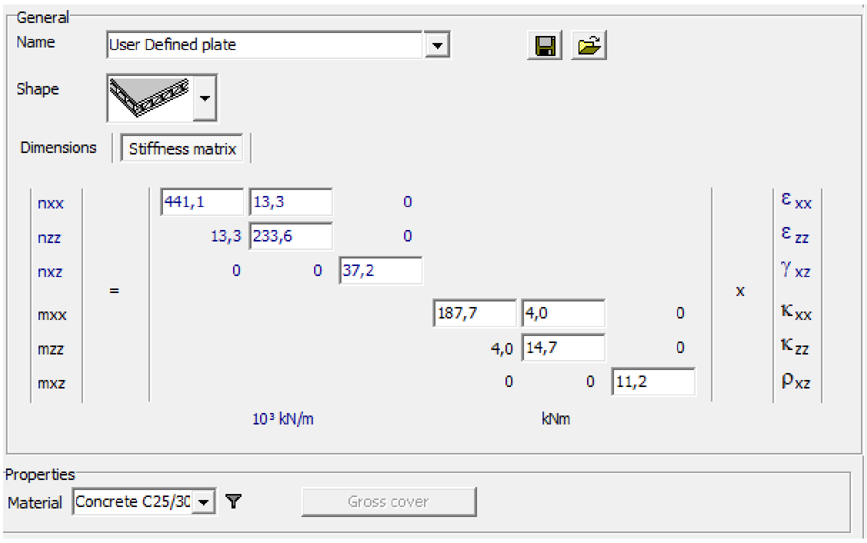
$$D_{66} = 11.2kNm$$

3.3.2. Enter the stiffness in Diamonds

To use the calculated stiffness in Diamonds:

- Select this shape 

.
- In the tab page *Dimensions*: assign a thickness and a material to the plate. Diamonds will use this to calculate the self-weight of the plate.
- Go to the tab page *Stiffness matrix*. Enter the values for the stiffness.



Notes:

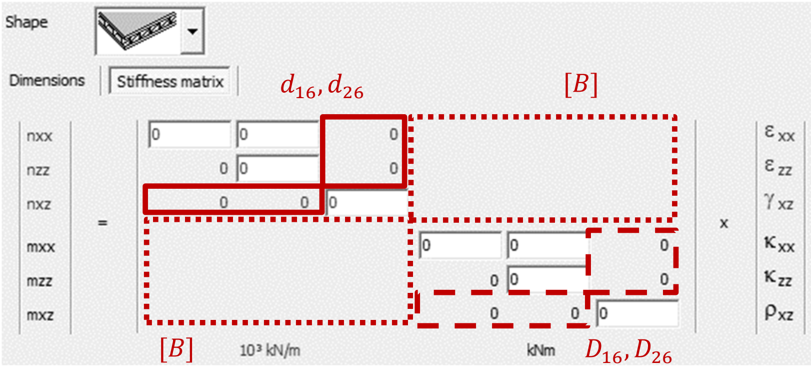
- Since the real shape of the plate is not known by Diamonds, no stresses nor reinforcement can be calculated.
  Pay attention when you calculate the stress: The strain varies linearly across the tickets. However, the stiffness properties (and thus the stresses) are discontinuous from one layer to the next!

Hypothetical 3-ply laminate | Strain variation | Young's modulus | Stress variation

Figure 5: Variation of strain and stress in a hypothetical 3-ply laminate

- The stiffness matrix in Diamonds assumes [1]:
  - $d_{16} = d_{26} = 0$
  - $D_{16} = D_{26} = 0$
  - $[B] = 0$

  This holds true, if all plies are orientated at 0° or 90° and when the composition of the plies is symmetrical.



- The calculated values can be verified with the tool from the eFunda website: http://www.efunda.com/formulae/solid_mechanics/composites/calc_ufrp_abd_go.cfm

3.3.3. Stiffness matrix for CLT plates

For the stiffness matrix of CLT plates, some additional simplifications and modifications are usually made [8-9]:

- The transverse stiffness $E_2 = 0$.
  $\Rightarrow$ the net cross-section (with $E = E_1 \neq 0$) is used to determine I and A for each direction (including only the CLT layers with fibre parallel to the considered direction of load-bearing).
- Transverse expansion (Poisson's) coefficients $\nu_{12} = \nu_{21} = 0$
- The in-plane shear stiffness d_{66} and the torsional stiffness D_{66} are reduced by using additional reduction factors k_s and k_D [8], which take into account the influence of longitudinal joints or longitudinal cracks of the lamellas.

Due to the small rolling shear stiffness of the transverse layers, the shear deformation of the plate usually needs to be considered. In Diamonds, the shear deformation cannot be considered explicitly, but it can be considered implicitly for simple cases such as simply supported plates loaded by uniform surface loads by using the **gamma method**. Using a gamma coefficient in the bending stiffness expression, the plate bending stiffness matrix elements D_{11} and D_{22} are reduced and thus the additional shear deformation is implicitly taken into account. More information about the gamma method and design examples can be found in [9-10].

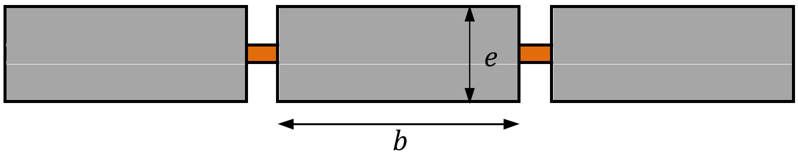
The stiffness matrices can be made or verified by using e.g. Calculatis from Stora Enso (https://calculatis.storaenso.com/) or CLTdesigner from holz.bau (https://www.cltdesigner.at/). These stiffness matrices explicitly consider the shear deformation by using an expanded plate stiffness matrix [11]. Thus the plate bending stiffness matrix elements D_{11} and D_{22} determined using these resources will not consider the shear deformation implicitly and will be different from those determined using the gamma method.

3.4. Torsional stiffness of plates bearing in one direction

The torisonal stiffness $D_{66.2}$ of a plate bearing in two direction equals [**2**]:

$$D_{66.2} = \frac{Ee^3}{12(1-\upsilon^2)} \cdot 0.5\,(1-\upsilon)$$

The torisonal stiffness $D_{66.1}$ of a plate bearing in one direction can be determined by considering this type of floor as a chain of beams (grey) with height e and width b. All beams are kept together by small strips (orange) with thickness $e/1000$.



The torisonal stiffness $D_{66.1}$ becomes (see Figure 4):

$$i_{xz} = \frac{e^3 \cdot b}{16}\left[\frac{16}{3} - \frac{3{,}36e}{b}\left(1 - \frac{e^4}{12b^4}\right)\right]$$

$$i_{zx} \approx 0$$

$$G = \frac{E}{2(1+\upsilon)}$$

$$D_{66.1} = 0{,}25 \cdot G \cdot (i_{xz} + i_{zx})$$

i_{xz} is the torsional constant I_t in the x'-direction smeared out over the width of one floor element.
i_{zx} is the torsional constant I_t in the z'- direction smeared out over the width of one floor element.

The torsional factor τ is the torsional stiffness of a plate bearing in one direction $D_{66.1}$ divided by the torsional stiffness of the plate bearing in two directions $D_{66.2}$.

$$\tau = \frac{D_{66.1}}{D_{66.2}} \cdot 100\ [\%]$$

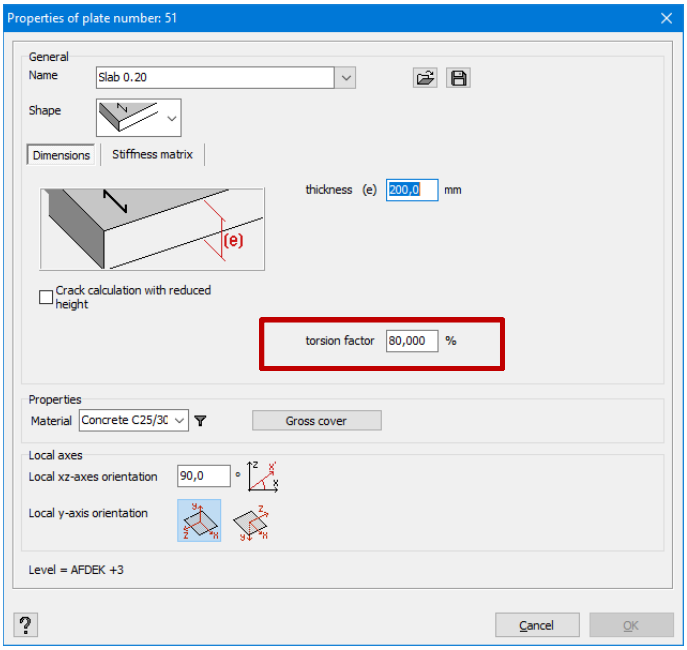
The table below describes the torsional stiffness of the plate bearing in one direction $D_{66.1}$ and the torsional factor τ for different width to thickness ratios b/e.

b/e [-]	$D_{66.1}$ $[kNm]$	τ [%]
1	3694	42%
2	6004	69%
3	6909	79%
4	7367	84%
6	7825	90%
10	8193	94%

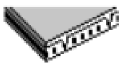
Figure 6: Torsional stiffness of a plate bearing in one direction $D_{66.1}$ and the torsional factor τ for different b/e ratios

From the table above we can conclude that for a plate bearing in one direction where the width of the strips is equal to the thickness e of the plate, 42% of the torsional stiffness of an isotropic plate is achieved.

When plates bearing in one direction are made as strips with thickness 20 cm and width 60 cm $\frac{b}{t} = \frac{60}{20} = 3$ we see that 80% of the torsional stiffness of an isotropic plate is achieved. In this case you should fill in 80% in the window with the plate properties:



Notes:

- If you want a plate bearing in one direction to behave like a beam, you should leave the torsional factor to 0%. If you want it to behave like a plate, the torsional factor will be > 0%
- For preslabs and hollow core slabs ( and ), Diamonds uses a width/ thickness-ratio $\frac{b}{t} = 3$ to determine the torsional stiffness.

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